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MULTIDIMENSIONAL SIGNAL RESTORATION AND BAND-LIMITED EXTRAPOLATION, II

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This technical report deals with signal restoration and extrapolation. First, we present some computer simulations to compare a number of existing algorithms for band-limited extrapolation. Then, we show a general Hilbert-spaces approach to iterative super-resolution which includes several well-knwon iterative procedures as special cases. Finally, we present some techniques for solving a general extrapolation problem. Applications to band-limited continuation are shown.

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Preface

This technical report consists of three parts. The central problem is the extrapolation of band-limited signals.

In part I, several existing algorithms for band-limited extrapolation are compared: Two-step procedures appeared to give better reconstructions and require less computing time than iterative algorithms.

In part II, five basic procedures for iterative restoration are unified using a Hilbert Space approach. In particular, all known iterative algorithms for extrapolation of band-limited signals are shown to be special cases of Bialy's iteration. ~~We~~ The authors also ~~obtain~~ obtained faster algorithms than that of Papoulis-Gerchberg.

In part III, the extrapolation problem is presented in a more general setting: Continuation of certain analytic functions. ~~We present~~ Presented are two ~~setps~~ step procedures for finding the continuation of these functions. Some new procedures for band-limited continuation are also discussed as well as the case in which the signal is contaminated with noise.

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Numerical Comparison of Several Algorithms for Band-Limited Signal Extrapolation

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ABSTRACT

We present some computer simulation results on the band-limited signal extrapolation problem. First, the performance of several existing algorithms are compared for the noise-free case. Then we describe some modifications of these algorithms for computing the extrapolation when the given signal is contaminated with noise. Computer simulation results for both the noiseless and noisy cases are included. From these results, the following preliminary conclusion could be drawn: Two-step algorithms appeared to give better reconstructions and require less computing time than iterative algorithms.

I. INTRODUCTION

The band-limited signal extrapolation problem was addressed by several authors ([1] - [10], among others). Some algorithms for computing the extrapolation were also given. But probably the most well-known algorithms in the engineering literature are those of [2], [3] and [6], [8]. Although some numerical examples were given in [6], a numerical comparison between both algorithms seems not to be available in the easily accessible literature. In this paper, the numerical performance of several existing algorithms are compared by means of computer simulation examples (Section II). Then some modifications of these algorithms are proposed for getting the extrapolation when the given signal is contaminated with noise.

Let us recall what is meant by band-limited signal extrapolation. Assume that g: $R \rightarrow C$ is an Ω-band-limited finite-energy signal, i.e.

$$\hat{g}(\omega) = 0, \quad \omega \notin [-\Omega, \Omega],$$

where $\hat{g}$ denotes the Fourier transform of g. If we are given a piece of g, g: $[-A, A] \rightarrow C$, we will be able to recover $g(x)$ for $x \notin [-A, A]$, because g is an analytic function. Band-limited signal extrapolation is the problem of computing $g(x)$ for all x from the known values $g(x)$, $x \in [-A, A]$. Several basic models relevant to the solution of the extrapolation problem were given in [1] (also [9]).

Probably the most well-known technique for solving the problem is given by the iterative procedure ([2], [3]):

$$g_0 = \text{initial} \quad \Omega\text{-band-limited approximation}$$

$$g_\alpha = \text{sinc}_\Omega * (J_{[-A,A]}g + (I - J_{[-A,A]}) \, g_{\alpha-1}) \quad\quad (1)$$

where $J_{[-A,A]}$ is the truncation operator to $[-A,A]$ and I denotes the identity. Some generalizations of this procedure were given in [4]. The numerical computation of (1) can be accomplished by means of the two following techniques:

(i) Implementing the convolution by using FFT

(ii) Sampling the iterative equation

$$h_0 = \text{initial guess}$$

$$h_u = h_{u-1} + \text{sinc}_\Omega * J_{[-A,A]}(g - h_{u-1}) \qquad (2)$$

Note that recursion (2) is equivalent to (1).

Technique (i) leads us to the following discrete recursion:

$$\gamma_{u+1}(j) = \frac{1}{2N} \sum_{m=-N}^{N-1} \alpha_u(m) \, . \, e^{2\pi i j m/N} , \qquad j \in [-N, N-1] \qquad (3a)$$

$$\alpha_u(m) = \begin{cases} 0 & m=-N, \; k_0 = [\frac{\Omega N \Delta}{\pi}] \leq |m| \leq N-1 \\ \\ \sum_{j=-N}^{N-1} \beta_u(j) e^{-2\pi i j m/N}, & |m| \leq k_0 \end{cases} \qquad (3b)$$

$$\beta_u(j) = \begin{cases} g(j), & |j| \leq [\frac{A}{\Delta}] \\ \\ \gamma_u(j), & j=-N \text{ or } [\frac{A}{\Delta}] \leq |j| \leq N-1 \end{cases} \qquad (3c)$$

In formulas (3b) and (3c) Δ denotes the distance between consecutive samples of g inside $[-A,A]$. It is to be noted that Δ should be chosen so that $[\frac{A}{\Delta}] = [\frac{\Omega N \Delta}{\pi}]$. The convergence of this procedure was shown in [5]. Some relationships

between the solution to (3a)-(3b)-(3c) and the solution to the extrapolation problem were given in [1]. In Appendix B, we will show that the limit of the iterative algorithm (3a)-(3b)-(3c) can be obtained by means of a certain two-step procedure ([6]). (A related discussion is given in [11]). Numerical comparison between both algorithms will be given in the next section.

On the other hand, technique (ii) originates the following iterative procedure:

$$S_u(j) = S_{u-1}(j) + \Delta \sum_{k \in [-A,A]} sinc_\Omega[(j-k)\Delta] \cdot (g(k\Delta) - S_{u-1}(k)) \qquad (4)$$

This recursion was shown to be convergent in [6]. It was also shown ([6]) that the limit of the procedure can be computed by means of a two-step algorithm. The relationship between this discrete technique and the solution to the extrapolation problem was discussed in [7]. In the next section, the solution given by this two-step procedure will be compared with the two techniques mentioned earlier.

If the piece of the signal g is contaminated with noise, the situation will be completely different. First, the extrapolation problem will not have any solution (unless the noise is also Ω-band-limited). Therefore, the problem has to be restated. Several attempts were made in this direction ([1],[6]).

Even though equations (1) and (2) are no longer convergent, the discrete iterative procedures (3a)-(3b)-(3c) and (4) are indeed convergent. However, the extrapolations obtained by means of the two-step algorithms are of poor quality in the presence of noise. This shows that some sort of stopping rule is necessary for the successful application of the iterative procedures in

order to prevent the noise from propagating too much into the reconstructions.

We also present some heuristic results obtained by modifying the two-step algorithms to cope with noise in the given part of the signal. This will be done in Section III. The numerical results obtained will be compared with an iterative Wienner-type procedure. The convergence of this iterative algorithm is proven in Appendix A.

II. THE NOISE-FREE CASE

In practice, the case in which g is not contaminated with any noise is not interesting. However, if no algorithm can perform well in the absence of noise, there will be no hope to solve the problem in cases where some noise is present.

As it was pointed out in the Introduction, three algorithms will be used for the extrapolation of g. Another algorithm for the noisy case will be presented in Section III. It can also be used for the noise-free case; this will be done in Section III.4. In our examples, the function g will be given by the formula

$$g(x) = \left(\frac{\sin \frac{\pi}{2} x}{\frac{\pi}{2} x} \right)^2 \cdot \cos \pi x, \quad x \in R$$

The Fourier transform of g is shown in figure 0. Note that g is band-limited to [-1,1].

We will use the following values for A; $A_1 = 1$, $A_2 = 1/2$ and $A_3 = 1/4$.

It is worth pointing out that there exists a fundamental difference between the case A_1, and the cases A_2 and A_3. When the function g is sampled

over x = (-1,1) (say, 32 samples) and DFT of these samples are used for
approximating g, the two peaks in the frequency space are still distinguished
(see Fig. 1a). On the other hand, this will not be the case for (-1/2,1/2)
and (-1/4,1/4). The corresponding plots are given in Figures 1(b) and 1(c).
All the extrapolations will be computed for x ∈ (-8,8).

II.1

In this section we will apply the discrete iterative procedure given by
equations (3a)-(3b)-(3c). For all the cases we will use 32 samples in the
known range of the signal g. Therefore, 32 unknown frequency values are to be
sought. For A = 1 the length of the DFT will be 256, for A = 1/2 it will be
512 and for A = 1/4 it will be 1024. Figures 2(a), 2(b) and 2(c) depict the
resolution of the Fourier transform obtained after 10 iterations for (-1,1),
(-1/2,1/2) and (-1/4,1/4), respectively. Figures 3(a), 3(b) and 3(c) show the
result obtained after 100 iterations. Several conclusions can be drawn from
these examples. For the case A = 1, some negative values of the spectrum have
been removed and the two peaks are also more clearly distinguished after 10
iterations. (Compare Figures 1(a) and 2(a)). However, the improvement
obtained after 100 iterations is not significant (see Figure 3(a)). For the
case A = 1/2 and A = 1/4, the peaks are not distinguishable after ten
iterations (see Figures 2(b) and 2(c)). This situation is the same for A =
1/4 when the number of iterations is 100 (see Figure 3(b)) and 1,000 (figure
4(b)). On the contrary, the situation improves for A = 1/2 when the number of
iterations is increased up to 1,000 (see Figure 4(a)).

II.2

As it was pointed out in Section I, the limit of the iterative procedure (3a)-(3b)-3(c) can also be obtained by means of a two-step procedure. This two-step procedure was also given in [6]. However, in [6], the relationship between this procedure and the algorithm (3a)-(3b)-(3c) was not discussed. Let L be the following (2n+1) x (2n+1) matrix,

$$L(k,h) = \frac{1}{M} \sum_{j=-n}^{n} e^{2\pi i j(k-h)/M}, \quad -n \leq k,h \leq n \qquad (5)$$

where M = 2N + 1. The matrix L is positive definite and therefore, we can always compute the solution x: $(x_h)_{-n \leq h \leq n}$ of the system of equations:

$$Lx = y \qquad (6)$$

The two-step procedure consists of first solving equation (6) when y = (g(kΔ), k = -n,...,n), and then computing the extrapolation as follows:

$$z_k = \sum_{h=-n}^{n} \frac{1}{M} \sum_{j=-n}^{n} e^{2\pi i j(k-h)/M} x_h \qquad (7)$$

where $-\infty < k < +\infty$. The extrapolation z_k is the limit of the iterative algorithm given by (3a)-(3b)-(3c). The proof of this fact is relegated to Appendix B. This way of proving the convergence of (3a)-(3b)-(3c) is simpler than that of [5] where non-expansive properties of certain operators were used. We have chosen an odd number of points so that L is real. In our examples we will use 33 points. Figure 5 depicts the Fourier transform of the extrapolations obtained by using this two-step procedure. Figure 5(b) shows a much better resolution for A = 1/2 than that obtained in Section II.1 (compare

Figure 5(b) with Figures 2(b), 3(b) and 4(a)). Figure 5(c) shows the result obtained for A = 1/4. The quality of the reconstruction is good, although some artifacts are present in the lower part of the peaks. This example also shows how slow the iterative technique given in Section II.1 is (Figure 4(b)) since it should converge to the same solution. We recall that the result of Figure 4(b) was obtained after 1,000 iterations.

II.3

We now consider another two-step procedure ([6],[8]). As we have pointed out in Section I, this two-step procedure will give us the limit of the iterative approach (4). This iterative procedure (4) and some of its generalizations were found to be very slow ([7]) and no numerical examples using them will be included here.

Let us consider the matrix $K \in R^{(2n+1) \times (2n+1)}$

$$k(i,j) = \frac{\sin \pi \Omega(i-j)}{\pi(i-j)}, \quad -n \leq i,j \leq n \tag{8}$$

We now solve the system of equations:

$$Kx = y, \tag{9}$$

where $y = (g(k\Delta), k = -n,\ldots,n)$. If $x = (x_h)_{-n \leq h \leq n}$ denotes the solution (which exists and is unique), the extrapolation is computed by means of the formula

$$\sum_{h=-n}^{n} \frac{\sin \pi \Omega \Delta(i-h)}{\pi(i-h)} x_h = s_i, \quad -\infty < i < +\infty \tag{10}$$

The Fourier transform of the extrapolations obtained by using this technique are shown in Figures 6(a), 6(b) and 6(c). It is worth pointing out that the resolutions obtained for all the cases are very good. For Δ = 1/4, the estimated Fourier transform is even better than that of Figure 5(c) since, for this algorithm, no artifacts are introduced in the reconstructions (see Figure 6(c)).

To conclude this section we would like to point out some differences between the two-step procedures (6)-(7) and (9)-(10). Both procedures are intended to provide approximations to the solution of the continuous extrapolation problem. However, the nature of the two approximations are completely different. The extrapolation provided by the two-step procedure (6)-(7) is a M-periodic discrete signal; z_k, $-\infty < k < +\infty$ which is band limited to $[-n,n]$:

$$\sum_{k=-N}^{N} z_k e^{-2\pi i k j / M} = 0, \quad |j| > n, \quad M = 2N+1$$

and $z_k = g(k\Delta)$, $k \leq n$. On the other hand, the extrapolation given by (9)-(10) is a finite-energy (and therefore non-periodic) sequence s_k: $-\infty < k < +\infty$, which is band-limited to $[-\Omega\Delta, \Omega\Delta]$, i.e.

$$\sum_{k=-\infty}^{+\infty} s_k e^{2\pi i k w} = 0, \quad |w| > \Omega\Delta$$

and $s_k = g(k\Delta)$, $|k| \leq n$. It is known that if $\Delta \to 0$, then the extrapolation s_k will approach the continuous extrapolation g ([7]). On the contrary, no similar result is known for the periodic extrapolation z_k. A more detailed discussion about four basic models for extrapolation which are related to the

two-step procedures can be found in [1].

III. THE NOISY CASE

In this section we will discuss several techniques for solving the extrapolation problem when the given signal is contaminated with some additive noise. The noise which will be used is white and uniform and its values will range on (-0.005, 0.005), on (-0.05, 0.05), and on (-0.5, 0.5). They will be called third digit, second digit and first digit noise respectively.

In what follows, we will compare the iterative procedure given by formulas (3a)-(3b)-3(c) applied to the noisy case with some heuristic modifications of the two-step procedures described in Sections II.2 and II.3. In Section III.4 these algorithms will be compared with a new iterative procedure designed to cope with noise.

III.1

We have mentioned in Section I that the iterative procedure 3(a)-3(b)-3(c) can also be applied to cases where the samples $g(k\Delta)$, $-n \leq k \leq n$ are corrupted with noise.

For A=1, the algorithm distinguishes the peaks well in the noise-free case (see figs. 1(a), 2(a) and 3(a)). Adding second digit noise to the known samples causes some artifacts in the reconstruction (see figures 7(a) and 7(b)). This is also the case when first digit noise is used (see figure 7(e)).

The solutions provided by the algorithm are almost the same as in the noise free case when third-digit noise is added to the known samples. This is

also the case when second-digit noise or first digit noise is added to the original signal g on $(-1/2,1/2)$ and $(-1/4,1/4)$. Figures 7(c) and 7(d) shows the result obtained after 500 iterations for $(-1/2,1/2)$ and $(-1/4,1/4)$ respectively when second digit noise is present in the samples. Figure 7(f) and 7(g) plot the corresponding Fourier transform after 500 iterations for first digit noise. Compare the results so·obtained with those of figures 3(b) and 3(c).

We know that the limit of this procedure coincides with the extrapolation obtained by using the two-step algorithm described in Section II.2 (see Appendix B). We have tried the two-step procedure given by formulas (6) and (7) when some noise is added to the samples $g(k\Delta)$, $-n \leq k \leq n$. The results are completely wrong due to the presence of noise. However, the iterative procedure seems not to be so sensitive to the noise. This is due to the very slow rate of convergence of the algorithm, which takes 1,000 iterations to build part of the peaks for $A = 1/2$ (figure 4(a)) and probably much more than that for $A = 1/4$ (figure 4(b)) when no noise is present. Therefore, the first thousand iterations will not be enough to get a reasonable approximation to the true extrapolation if some noise is added to the given samples. It is clear that any attempt to speed up the convergence of this iterative procedure will also propagate the noise faster than the present algorithm.

III.2

In this section, we modify the two-step procedure given by formulas (6) and (7) to cope with noise. The modification consists of adding a positive number γ to the main diagonal of the matrix L, obtaining a new matrix Γ, then solving the system of equation $\Gamma x = y$ and using formula (7) to get the

extrapolation.

The motivation for this modification is two-fold. It is well known that L is an ill-conditioned matrix; therefore small perturbations introduced in $g(k\Delta)$, $-n \leq k \leq n$ may produce large changes in the solution x_h, $-n \leq h \leq n$, and this may overwhelm the extrapolation: z_k, $-N \leq h \leq N$. If we add some small positive number to the diagonal, the matrix will become better conditioned and, therefore, the solution of the system of equation (6) will be a more stable problem.

The other reason for such modification is that the two-step procedure was shown to be the best estimation to the extrapolation problem when the noise to signal ratio is added to the diagonal of the matrix K given by formula (8) (see [6]). It is obvious that K and L are different. However, some related optimality property might be proven when L is used instead of K.

Figures 8(a), 8(b), 8(c) and 8(d) show the results obtained for the case A = 1 by means of this regularization technique. Third-digit noise has been added to the signal. Even though the spectra are different, the sensitivity of the reconstruction with respect to the parameter γ is not large.

Figures 9(a), 9(b), 9(c), 9(d) shows the results obtained for A = 1 when second-digit noise is added to the given signal. In this case, the sensitivity of the reconstruction with respect to λ is more evident.
Figures 9(e), 9(f) corresponds to the first digit noise case. It is worth noting that the case given in 9(e) will be a reasonable reconstruction if the available information about the exact band-width is used before plotting the Fourier transform.

Figures 10(a), 10(b) and 10(c) show the spectra of the reconstructed signal for A = 1 when second-digit noise is used. The result obtained is not of good quality. However, some more a priori information about the true spectrum will be of great help. For instance, a positivity constraint applied to figures 10(a) or 10(b) will provide a much better result. This is also the case when first digit noise is used. Figures 10(d), 10(e) show the reconstruction for this case.

Figures 11(a), 11(b) and 11(c) show the results obtained for the case A = 1/4 when second-digit noise is used. It is clear that the sensitivity to the parameter λ is much more critical. This is also the case when first digit noise is added to the signal (see fig. 11(d), 11(e)). Once more, a positivity constraint will provide a good reconstruction if λ=0.000003 is used (fig. 11(a)). Figure 12 depicts the results obtained when third-digit noise is added. We would like to emphasize that if positivity of the Fourier transform is used as a priori information, that is to say, we set to zero all negative values in the reconstructed Fourier transform, the results are better than those obtained when no extrapolation is performed. For instance, fig. 11(f) depicts the Fourier transformation of the signal plus noise in (-1/4,1/4) when no super-resolution is tried. Compare this result with that of fig. 11(d) improved when positivity information is incorporated in the spectrum.

III.3

We have pointed out in III.2 that the same regularization technique can be applied to the two-step procedure given by formulas (9) and (10) (see Section II.3). Some motivations for such techniques can be found in [6]. If we denote by $\tilde{K}$ the matrix $K + \lambda I$, where I is the identity, the technique

consists of solving the system of equations $K x = y$ (where $y = (g(k\Delta))$ $k = -n,\ldots, n$) and using equation (10) to obtain the extrapolation. Figure 13(a) shows the results obtained with this technique for the case $A = 1$ and when third-digit noise is used. Figures 13(b) and 13(c) show the results obtained for two different values of λ, where second-digit noise is added to the signal. The corresponding plots for first-digit noise are shown in fig. 13(d), 13(e).

Figures 14 and 15 show the reconstruction obtained when third-digit, second-digit and first-digit noise is used for the case $A = 1/2$. Figures 16 and 17 present the corresponding results for the case $A = 1/4$. For this technique, we have shown the plots corresponding to the best results obtained.

Several conclusions can be drawn from these examples. For $A = 1/4$ and $A = 1/2$, when first-digit noise is used, it is seen (figs. 15(c), 15(d), 17(c), 17(d) and 17(e)) that none of the values used for λ provide a reasonable result. In this case, the main conclusion is that the reconstructed Fourier transforms are of very poor quality. (The case $A = 1/4$ is even worse than $A = 1/2$) For $A = 1$ and first-digit noise, the results look much more encouraging. Taking into account the amount of noise introduced in the observation, we conclude that the reconstructions are acceptable. As we have remarked above they will improve if some positivity constraint is used to remove the negative artifacts (see fig. 13). Another important observation is that the results obtained for $A = 1/4$, $A = 1/2$ are much better when the technique given in III.2 is used instead. Comparing fig. 10(e) with fig. 15(d) and fig. 11(d) with fig. 17(c), 17(d), 17(e) we conclude that the two-step procedure used in III.2 suits this numerical simulation better than that of section III.3 when noise becomes high enough. In case of lower noise we see that the best values

for λ need to be smaller as the observation part of the signal shrinks. In addition, the sensitivity of the reconstructed Fourier transform in terms of λ increases as the length of the known part decreases; for example, figures 17(a) and 17(b) show that the result may change drastically. It is also clear that a smaller λ results in more positive and negative overshoots and ringing. Again, some a priori information may help improve the reconstruction. This will be the case if a positivity contraint is used because the negative artifacts in figures 13(d), 14(b), 14(c), 15(b), 15(c), 15(d), 16(a), 16(b) and 17(b) will then be removed.

III.4

In this Section, we present a new iterative technique for the noisy case. We will make use of the Periodic Discrete Prolate Spheriodal Sequences (P-DPSS) (see Appendix A).

Let $f(m)$ be a periodic bandlimited signal of period 2N-1,

$$\hat{f}(k) = 0 \text{ for } U < |k| \leq N-1 \tag{11}$$

where $\hat{f}(k)$ is the DFT of $f(m)$, and U<N.

Let

$$y(m) = f(m) + n(m), \text{ for } 0 \leq m < D \tag{12}$$

$y(m)$ is the observed sequence of length D. The operator T is defined as

$$T \ x(m) = \begin{cases} x(m); & 0 \leq m < D \\ \\ 0 & ; \text{ otherwise} \end{cases} \tag{13}$$

and the operator B is defined as the following

$$B \; x(m) \; = \; IDFT \begin{cases} x(m); & |k| < U \\ \\ 0 & ; \quad \text{otherwise} \end{cases} \qquad (14)$$

Also for convenience, we define

$$(x(m) \; , \; y(m))_N \; = \; \sum_{m=0}^{N-1} x(m) y(m) \qquad (15)$$

$$(x(m) \; , \; y(m))_D \; = \; \sum_{m=0}^{D-1} x(m) y(m) \qquad (16)$$

For the given noisy observation $y(m)$, we can expand it in terms of $\phi_k(m)$ as follows

$$y(m) \; = \; \sum_{k=0}^{D-1} b_k \; \phi_k(m) \; , \; m \in [0, D-1] \qquad (17)$$

where

$$b_k \; = \; 1/\lambda_k \; (y(m), \; \phi_k(m))_D$$

$$= \; 1/\lambda_k \; (f(m), \; \phi_k(m))_M + 1/\lambda_k \; (n(m), \; \phi_k(m))_D$$

$$= \; a_k + 1/\lambda_k \cdot n_k$$

and

$$a_k \; = \; (f(m), \; \phi_k(m))_N$$

$$n_k \; = \; (n(m), \; \phi_k(m))_D$$

We wish to find a set of c_k's such that

$$f_e(m) = \sum_{k=0}^{D-1} c_k b_k \phi_k(m) \; ; \; 0 \le m < N \tag{18}$$

minimizes the mean square error $E \; (f-f_e, \; f-f_e)_N$.

Using the orthogonality properties of ϕ_k's, we can write the mean square error as

$$E \left(\sum_{k=0}^{K-1} (a_k - c_k b_k)^2 \right)$$

minimization of the mean square error leads to

$$c_k = \frac{E \; (a_k b_k)}{E \; (b_k^2)} = \frac{\sigma_{a_k}^2}{\sigma_{n_k}^2 + \sigma_{n_k}^2 \frac{1}{\lambda_k^2}} = \frac{1}{1 + \frac{\sigma_{n_k}^2}{\lambda_k^2 \sigma_{a_k}^2}} \tag{19}$$

where we have assumed that the noise and the signal are independent.

Equation (19) is similar in form to the familiar Weiner filtering. In practice, $\sigma_{a_k}^2$, $\sigma_{n_k}^2$, and λ_k^2 are difficult if not impossible to obtain. A possible approximation is to assume

$$\frac{n_k^2}{\sigma_{a_k}^2} = \mu = \text{constant} \tag{20}$$

$$c_k = 1/(1 + \frac{\mu}{\lambda_k^2}), \tag{21}$$

Even with these approximations, direct solution still require solving for λ_k's. Fortunately, there exists an iterative algorithm which converges to the function given by Equation (18) when c_k is given by (21).

Iterative Algorithm:

Let $f_0(m) = 0$

$$f_{u+1}(m) = (1-\mu)f_u(m) + BT\,[BT[y(m) - f_u(m)]] \tag{22}$$

then
$$\lim_{u\to\infty} f_u(m) = \sum_{k=0}^{D-1} 1/(1 + \frac{\mu}{\lambda_k^2})\, b_k \phi_k(m) \tag{23}$$

The proof of its convergence is relegated to Appendix A. It is clear that if there is no noise present, we obtain another algorithm for the noise-free case by setting $\mu = 0$ in formula (22).

Figures 18(a), 18(b) and 18(c) show the result obtained with $\mu = 0$ after 200 iterations. In fig. 18(a), the Fourier transform of the extrapolation for $A = 1$ is shown. The result can be judged as good. However, figures 17(b) and 17(c) show that the procedure suffers the same drawback as the algorithm given in II.1: The number of iteration required to distinguish the peaks is enormous.

Figures 19(a) and 19(b) shows the performance of the algorithm for $(-1,1)$ when second-digit noise is added to the given signal. The value used for μ was 0.0. Figures 20(a) and 20(b) shows the result obtained by the same procedure when $\mu = 0.01$. It is clear that the new value chosen for μ eliminates the artifacts of figure 19(b) after 200 iterations (figure 20(b)).

Figure 21 show the results obtained by using the procedure for $A = 1/2$ where second-digit noise is used. Once more, the different performance between $\mu = 0$ and $\mu \neq 0$ is clear. Figure 21 (b) shows that the artifacts of Figure 21 8a) have been eliminated. However, the peaks are less distinguishable. The results obtained for the case $A = 1/4$ are similar to those of section III.4 and they will not be included here.

IV. CONCLUSIONS AND DISCUSSIONS

The numerical performance of several algorithms was compared. Two basic assumptions were made: the continuous cut-off frequency of the given signal is known exactly, and the Fourier transform of the signal was supposed to be real. In the numerical examples given in Sections II and III the imaginary part of the reconstructed Fourier transform was negligible. It turns out from these numerical examples that the non-iterative techniques (two-step procedures) produce better results than those provided by the iterative procedures. More numerical examples are in order to verify or disprove this conclusion. Another important aspect of the algorithms which has to be compared is the number of operations involved in the procedures. If $2n$ is the number of given samples of g in $(-A,A)$ both two-step procedures need $0(n^2)$ operations. This is because the matrix involved in the system of equations (6) and (9) are Toeplitz and therefore $0(n^2)$-algorithms are known for solving the system. On the other hand, at every step of the iterative procedures (3a)-(3b)-(3c) and (22) at least one FFT is needed. The length of this FFT is $N=cn^2$ c denotes a constant; this is because the following equation has to be satisfied: $\left[\frac{\Omega AN}{n2\pi}\right]=n$. Therefore, we will have $0(n^2 \log n)$ operations per step. This analysis shows that the two-step procedures are less expensive in terms of arithmetic computation time. However, the performance of the two-step procedures for the noisy case depends on choosing the correct parameter λ. Therefore, the relationship between the optimal λ and the noise has to be further studied.

The sensitivity of all these algorithms to changes in the cut-off frequency has to be investigated because the exact value of the highest frequency might not be available in practice.

Acknowledgement

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Appendix A

To establish notation, some properties of Periodic Discrete Prolate Spherodial Sequence (P-DPSS) [6] are listed below.

For $D > 0$ and $U > 0$, we can find

$$\varphi_0(m), \phi_1(m), \ldots, \phi_{D-1}(m), m \in Z$$

and $\quad \lambda_0, \lambda_1, \ldots, \lambda_{D-1} \quad$ such that

(i) $BT\phi_k(m) = \lambda_k\phi_k(m)$; $0 \leq m < N$

$$k = 0,1, \ldots, D-1$$

(ii) $\phi_k(m+N) = \phi_k(m)$

(iii) $(\phi_k(m), \phi_j(m))_N = \delta_{k,j}$

$\quad (\phi_k(m), \phi_j(m))_M = \lambda_k \, \delta_{k,j}$

(iv) $B \, \phi_k(m) = \phi_k(m)$, ϕ_k is bandlimited

(v) $\phi_k(m)$, $k = 0,1, \ldots, D-1$ form a basis in the vector space of sequences of length D.

B and T are defined by equation (13) and (14) respectively.

We will now prove the convergence of the procedure (22),

Section III.A.

Let $\quad f_u(m) = \displaystyle\sum_{k=0}^{D-1} d_{k,u} \, \phi_k(m)$, $0 \leq m < N$. $\hspace{2cm}$ (A.1)

by substituting Equation (A.1) and Equation (17) into Equation (22), we have

$$d_{k,u+1} = (1-\mu)d_{k,u} + \lambda_k^2(b_k - d_{k,u}) \tag{A.2}$$

$$= (1 - (\mu+\lambda_k^2))d_{k,u} + \lambda_k^2 b_k$$

so

$$d_{k,u} = \frac{\lambda_k^2 b_k}{\lambda_k^2+\mu}(1-(1-(\lambda_k^2 + \mu))^u) \tag{A.3}$$

chosing $\mu > 0$ such that $|1-(\lambda_k^2 + \mu)| < 1$, or $\mu + \lambda_k^2 < 2$

we have

$$\lim_{u\to\infty} f_u(m) = \sum \frac{\lambda_k^2 b_k}{\lambda_k^2+\mu} \phi_k(m), \text{ which proves Equation (23).}$$

Appendix B

We prove that the two-step procedure given b formulas (6) and (7) in Section II.2 provides the same extrapolation as the iterative technique (3a)-(3b)-(3c).

We know that the matrix L given by (5) is symmetric and positive definite. It is also easy to prove that all its eigenvalues are less than 1. Therefore, the solution of the system of equations (6) can be computed as follows:

$$X^0 = 0$$

$$X^{u+1} = X^u + (y - L X^u) , u \geq 0 \tag{B.1}$$

If we also denote by L: $L(k,h)$ the matrix defined by (5) when $k \varepsilon (-\infty,+\infty)$, $h \varepsilon [-N,N]$ and if we apply L to both sides of equation (B.1), we will obtain

$$y^0 = 0$$

$$y^{u+1} = y^u + LT(y - y^u) \tag{B.2}$$

where $T(y - y^u)(m) = 0$ if $m \not\varepsilon [-n,n]$ and $(y-y^u)(m)$ if $m \varepsilon [-n,n]$. It is easy to see that $Ly^u = y^u$, for all $u \geq 0$. Therefore, equation (B.2) can be written as

$$y^0 = 0$$

$$y^{u+1} = L(y^u + Ty - Ty^u) = L(Ty + (I-T)y^u) \tag{B.3}$$

where $(I-T)y^u(m) = 0$ if $m \varepsilon [-n,n]$ and $y^u(m)$ if $m \not\varepsilon [-n,n]$.

It is now a simple exercise to verify the equivalence of the procedures (B.3) and (3a)-(3b)-(3c).

Figure 1.a. DFT of 33 samples of $g(x)$ in $(-1,1)$.
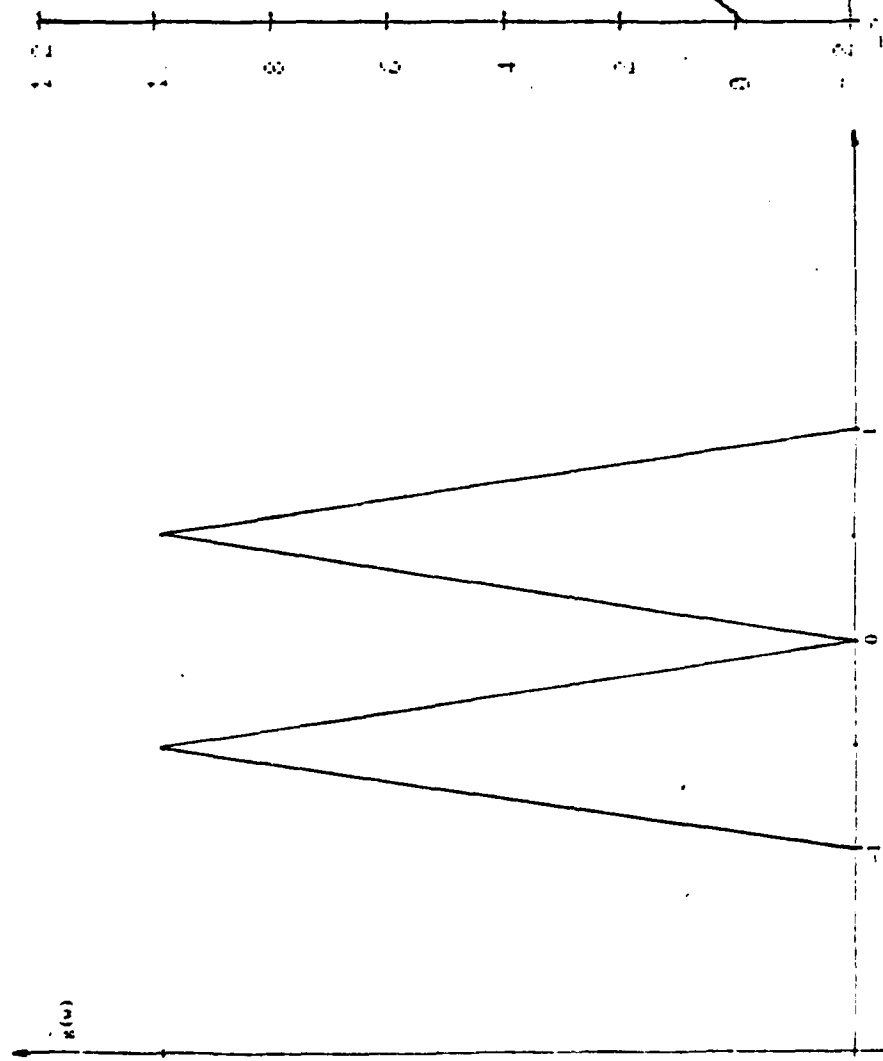
Figure 0: Fourier Transform of the Ideal signal.
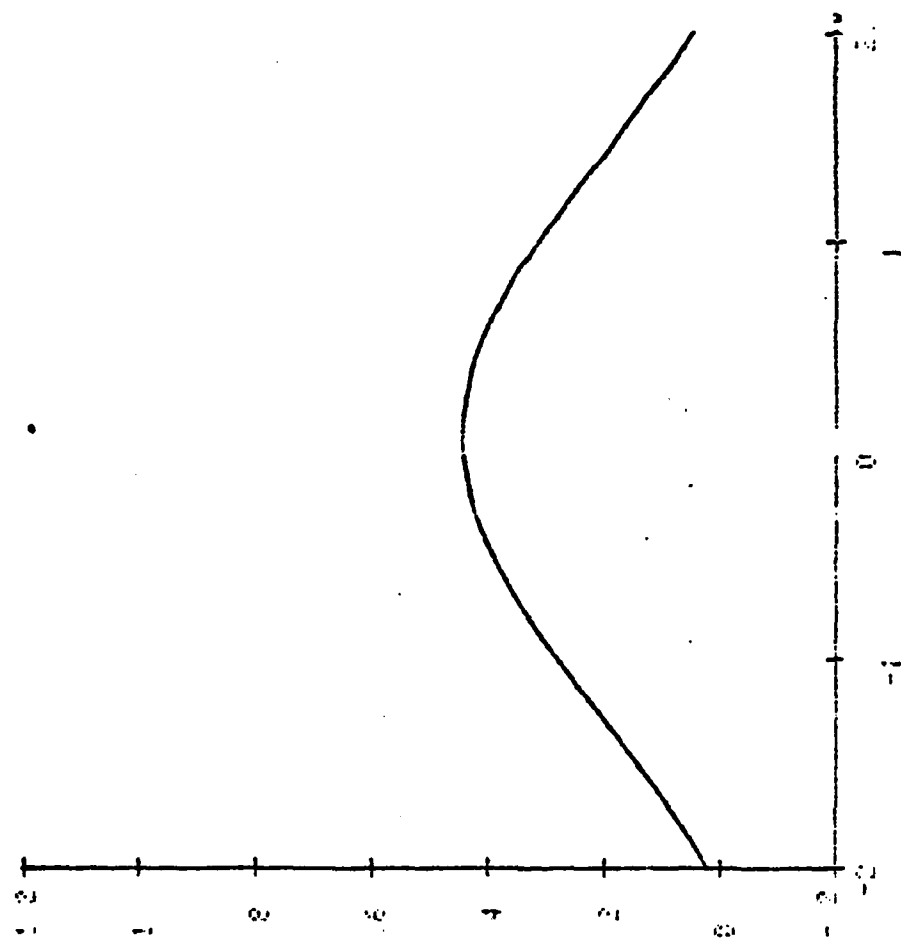
Figure 1.c. DFT of 33 samples of g(x) in $(-\frac{1}{4},\frac{1}{4})$.

Figure 1.b. DFT of 33 samples of g(x) in $(-\frac{1}{2},\frac{1}{2})$.
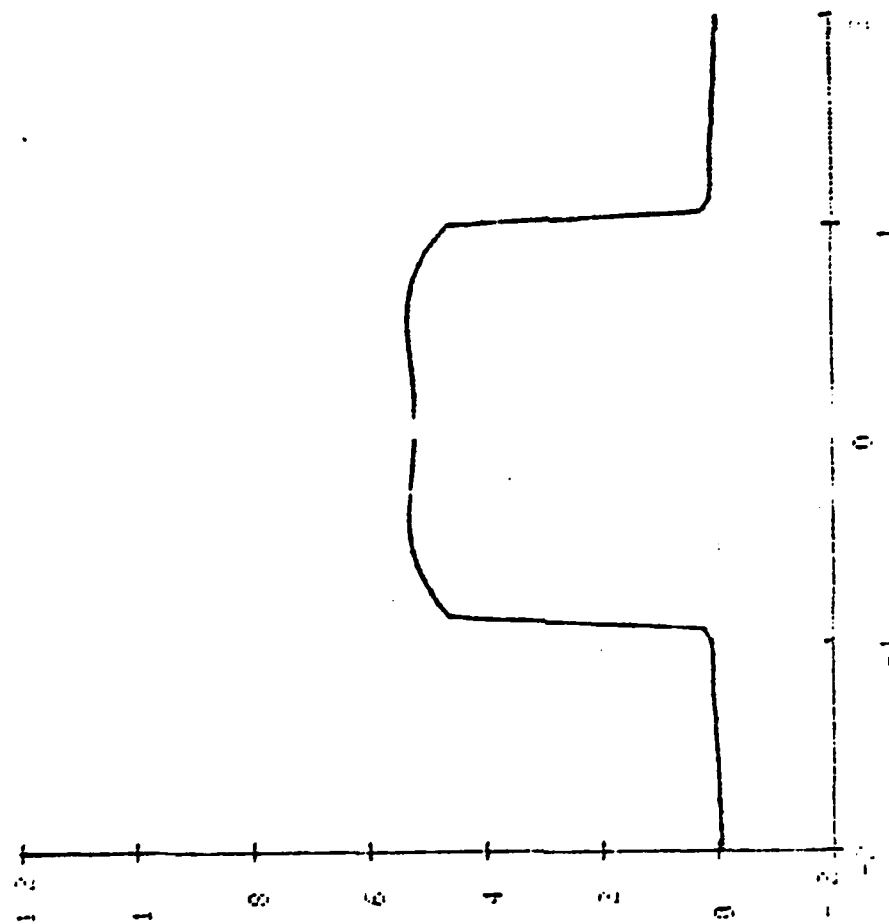
Figure 2.b. 10 iterations. λ = 3.
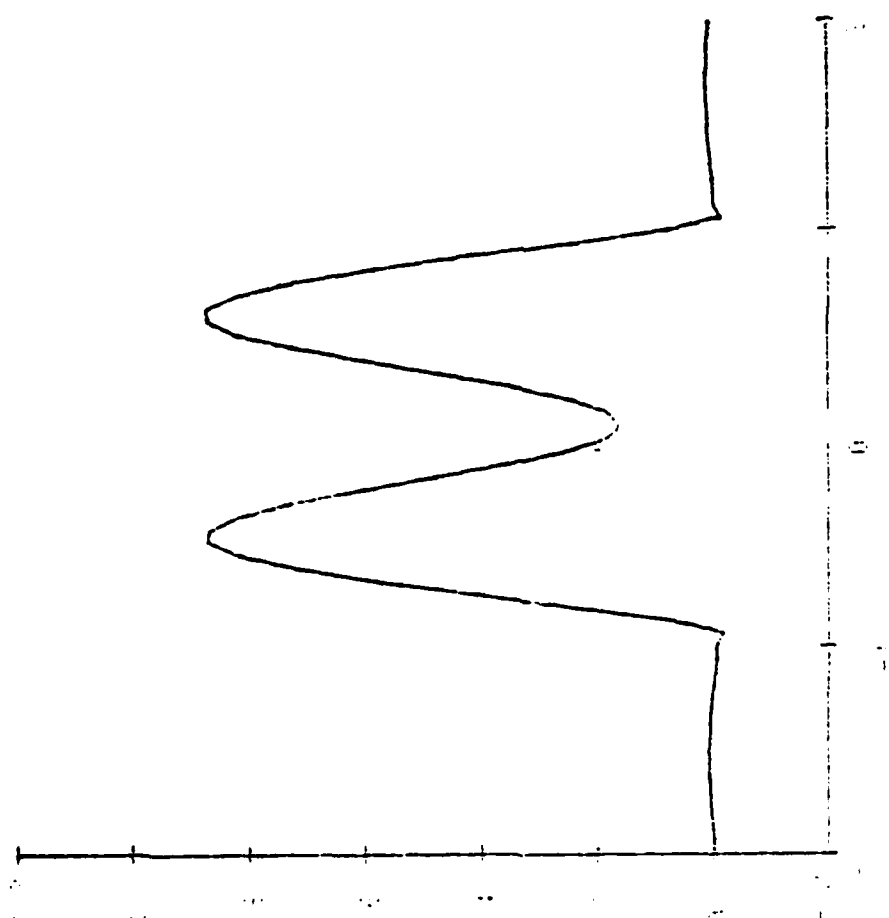
Figure 2.a. 10 iterations. λ = 1.
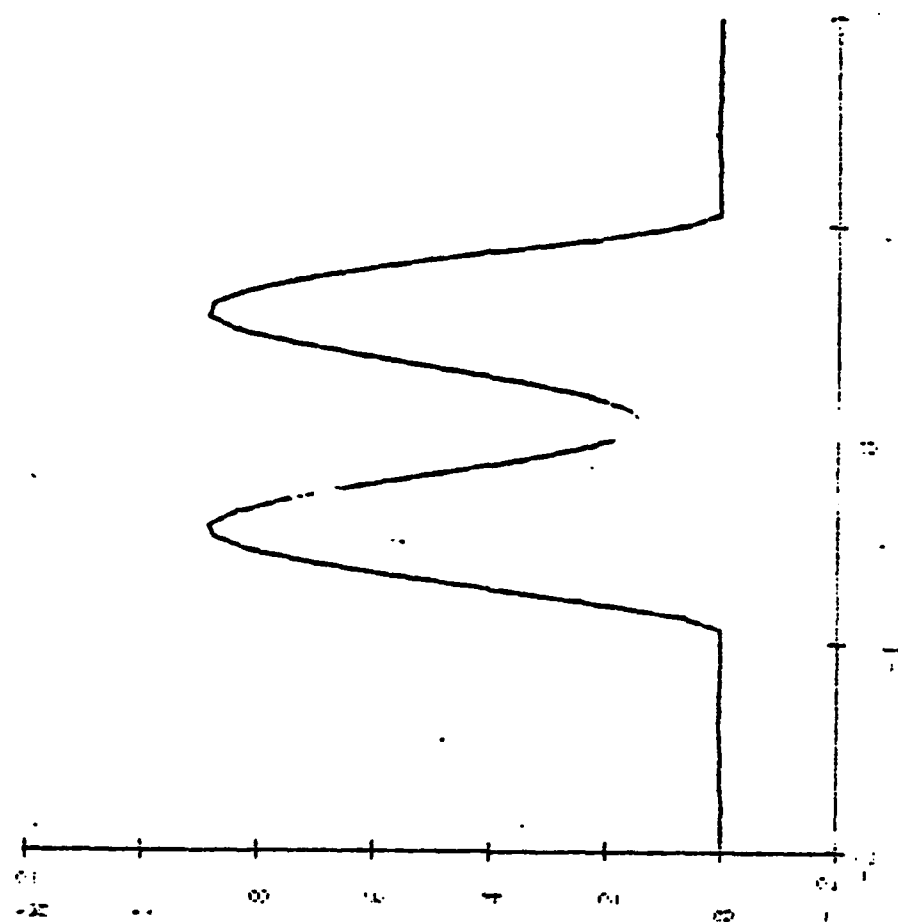
Figure 3.4. 100 iterations. A 1.

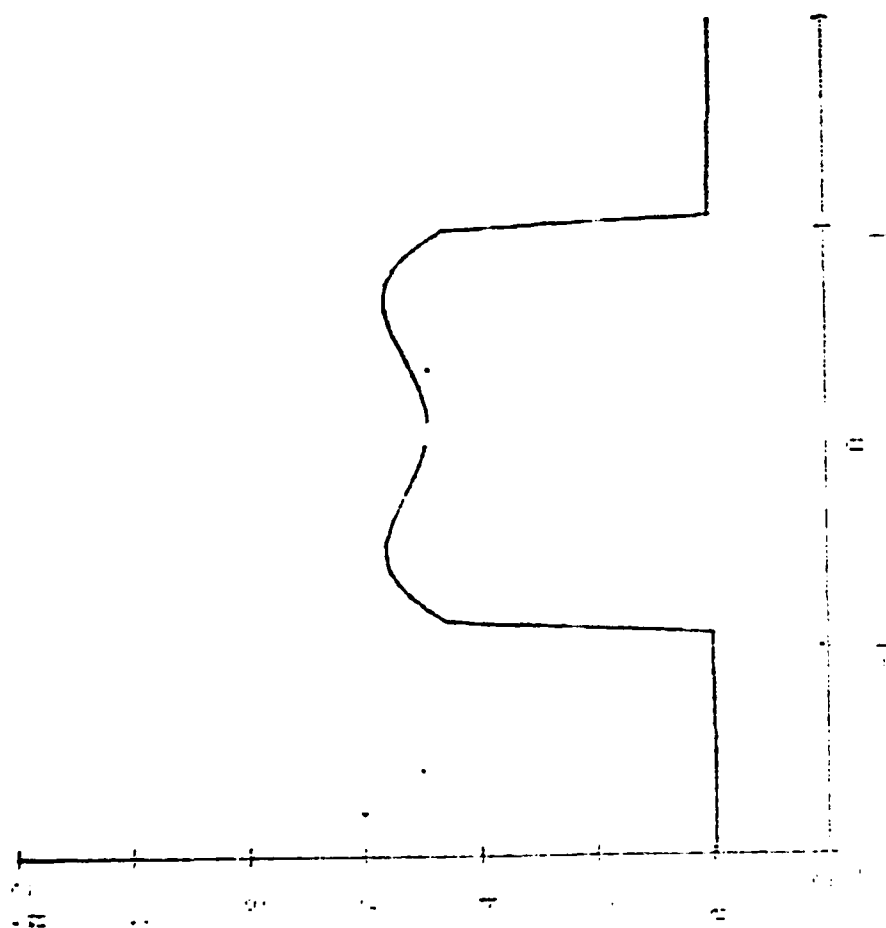Figure 2.4. 10 iterations. A 1.
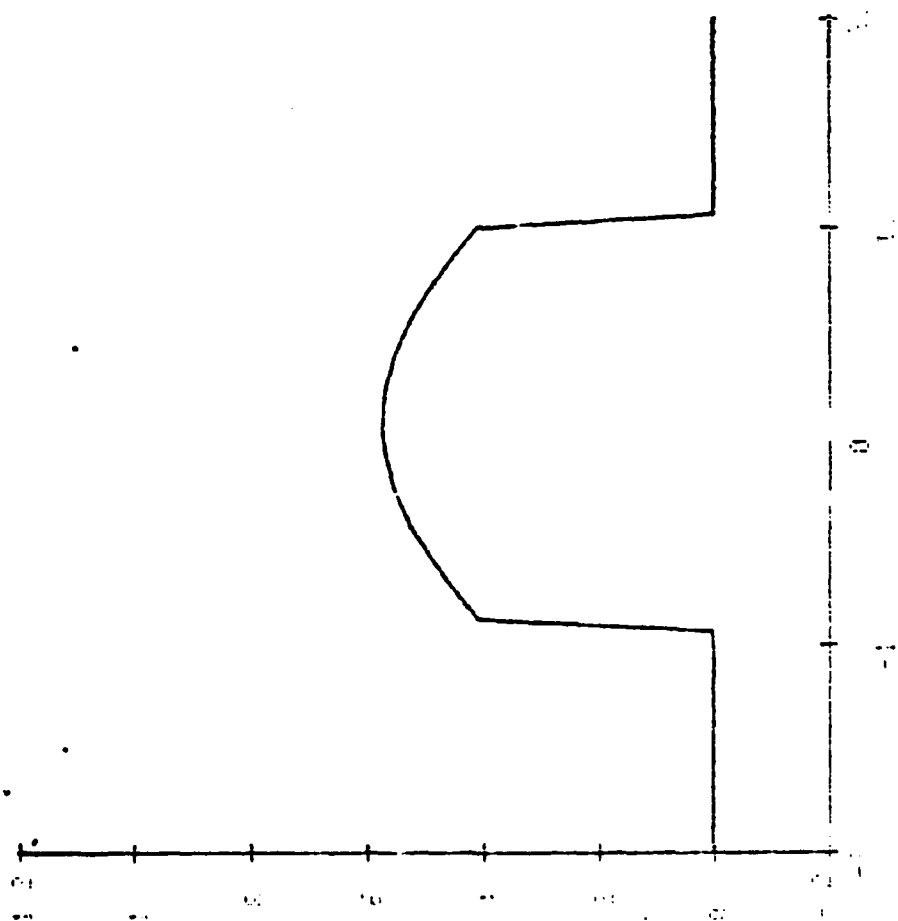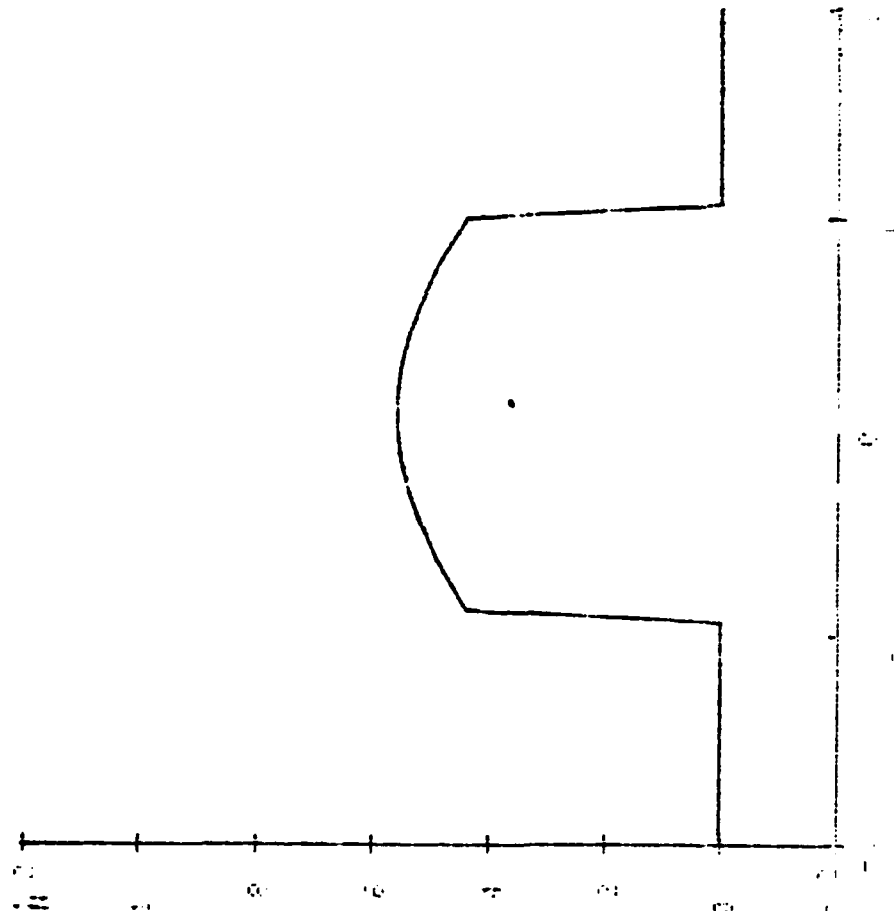
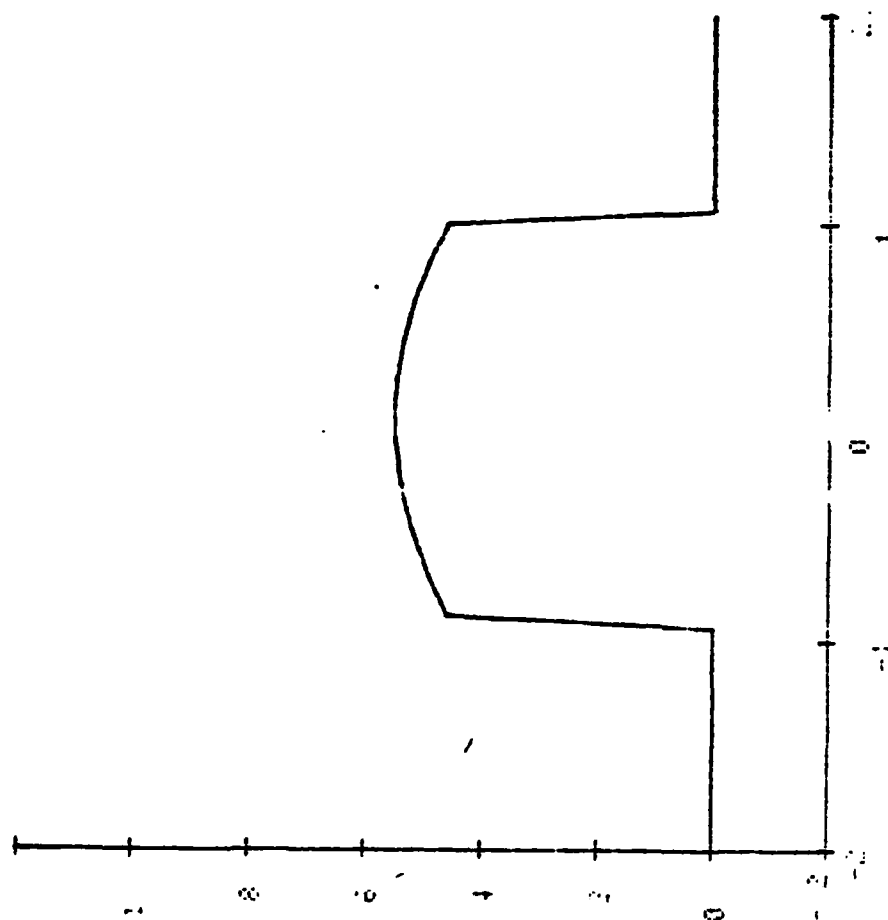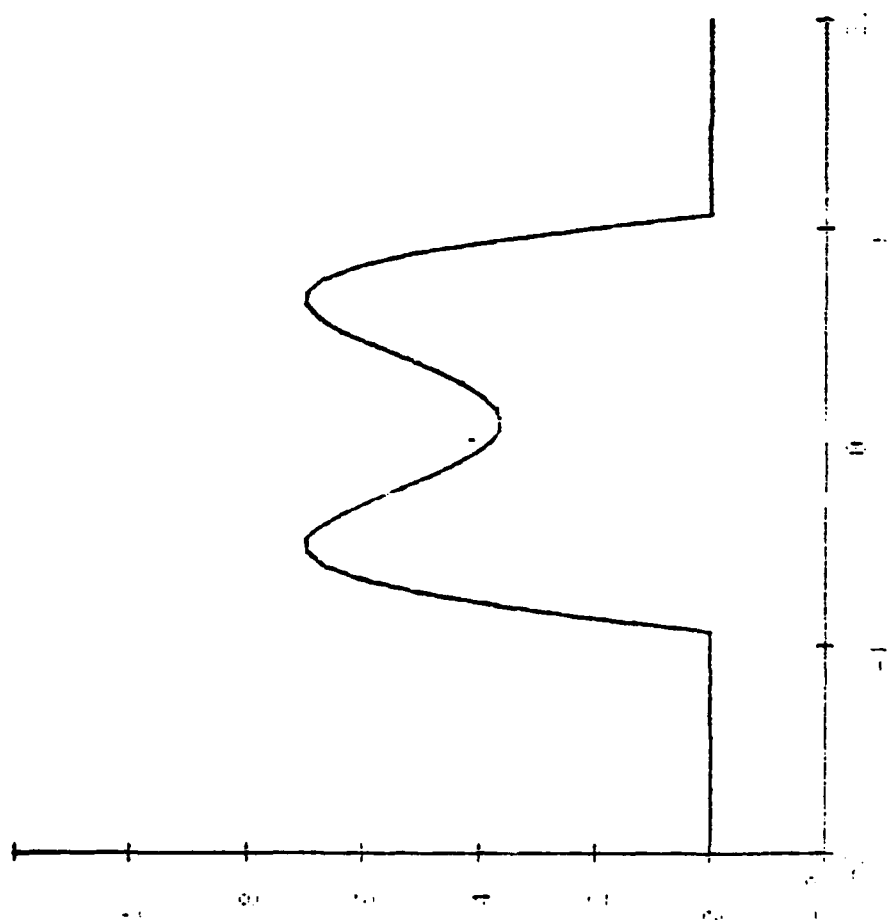Figure 1.c. 100 iterations. A - U.

Figure 1.b. 100 iterations. A - U.

Figure 4.b. 1000 iterations. A Ꞵₐ.

Figure 4.a. 1000 iterations. A Ꞵₐ.

Figure 5.b. Two-step procedure I. A = 5.
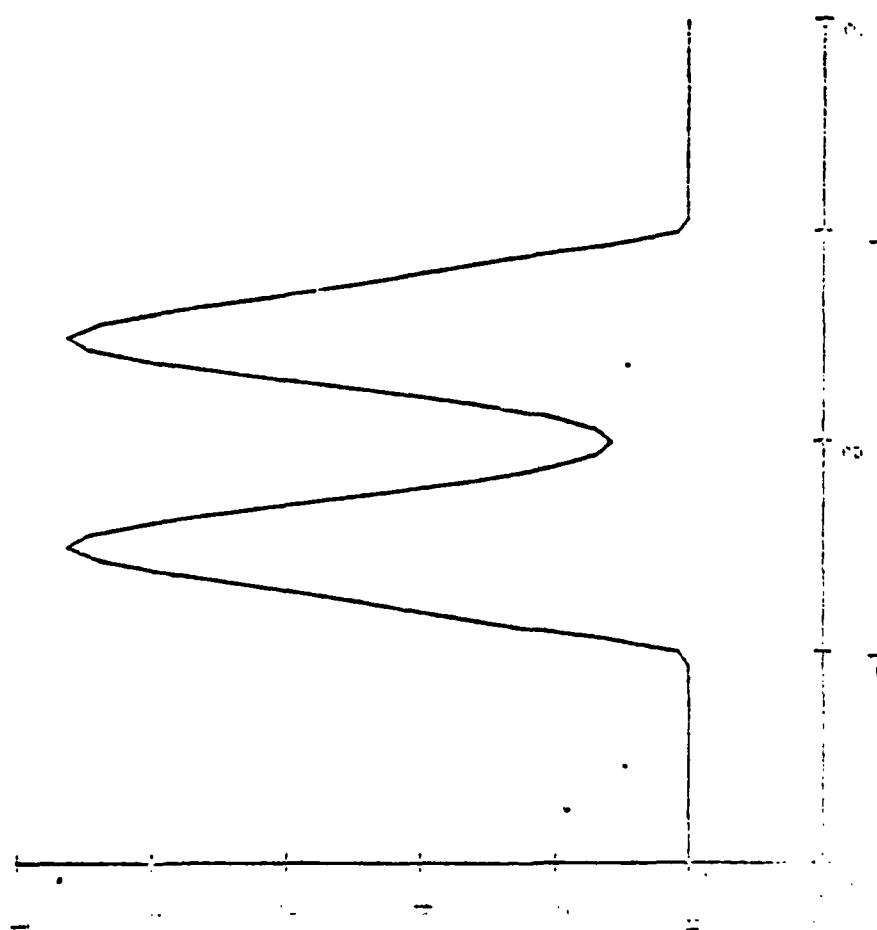
Figure 5.a. Two-step procedure I. A = 1.

Figure 6.d. Two-step procedure 2. A = 1.
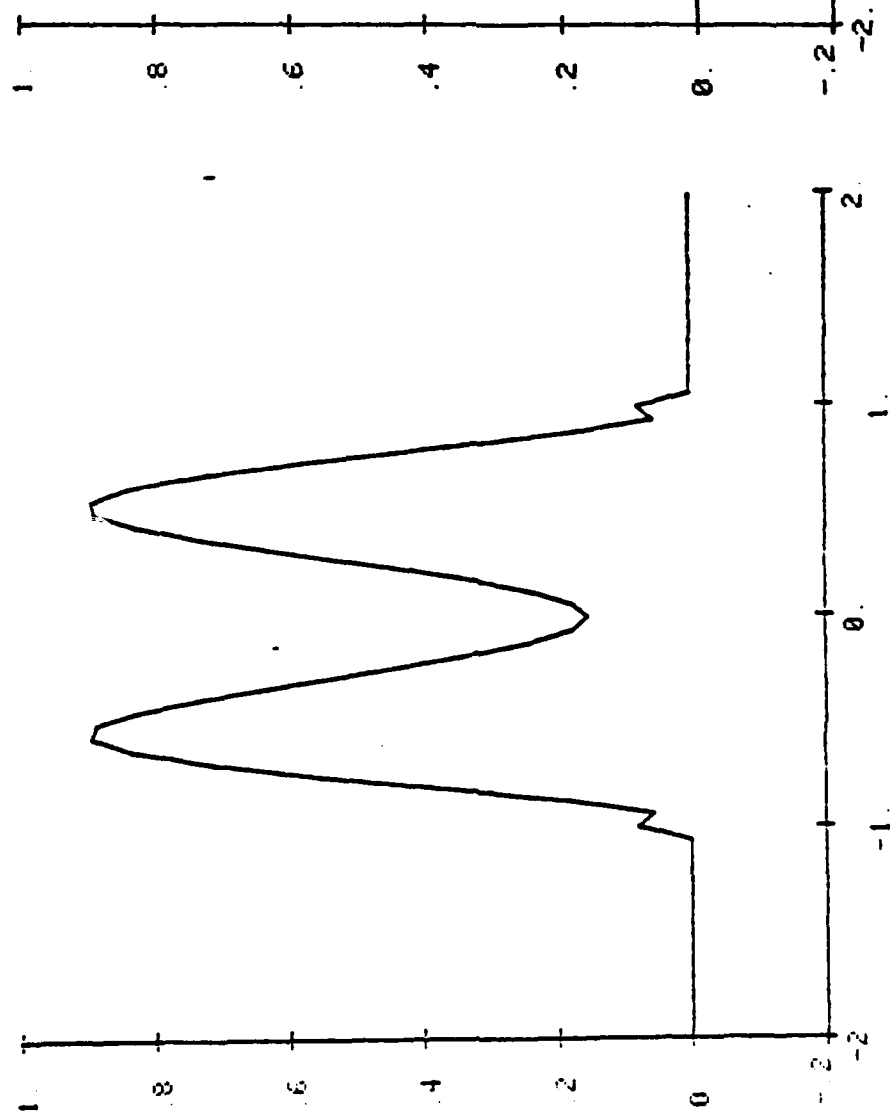
Figure 5.c. Two-step procedure. A = 1.

I.33

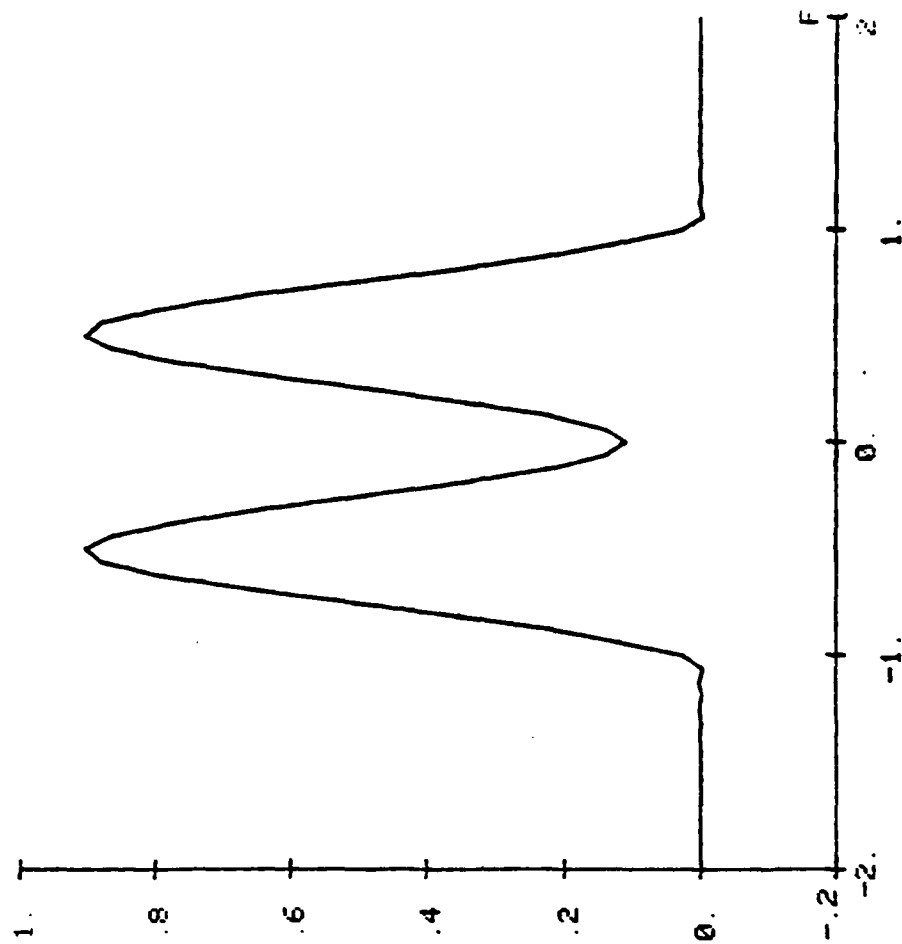Figure 6.c. Two-step procedure A
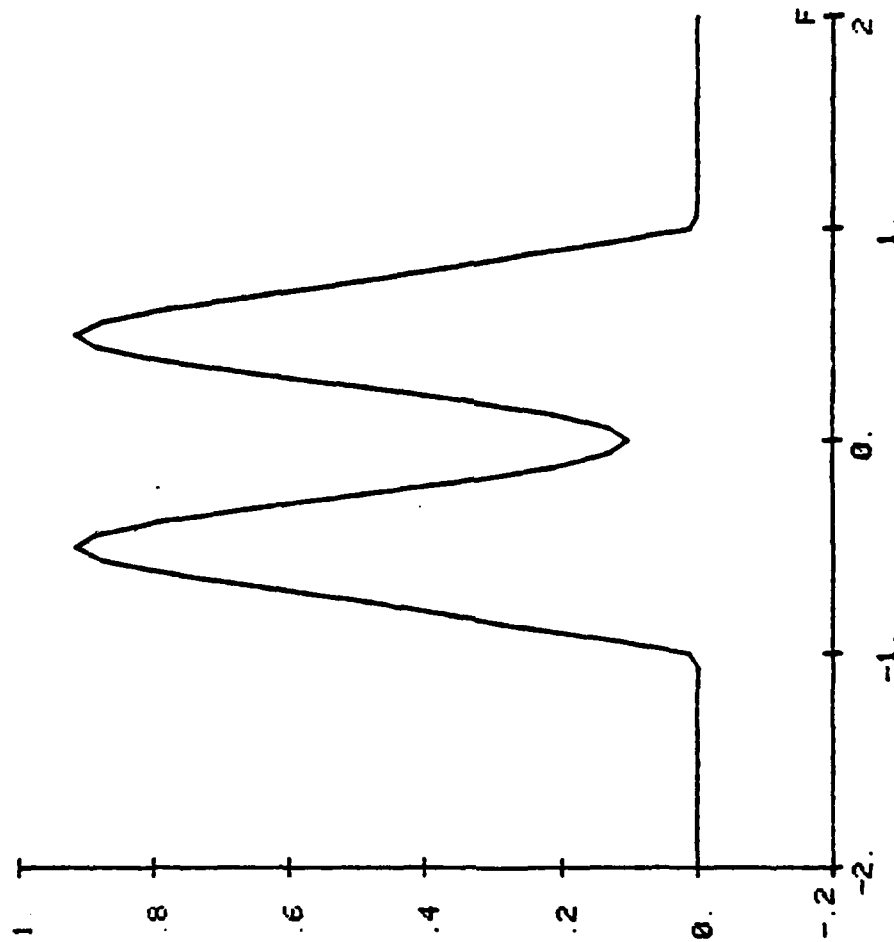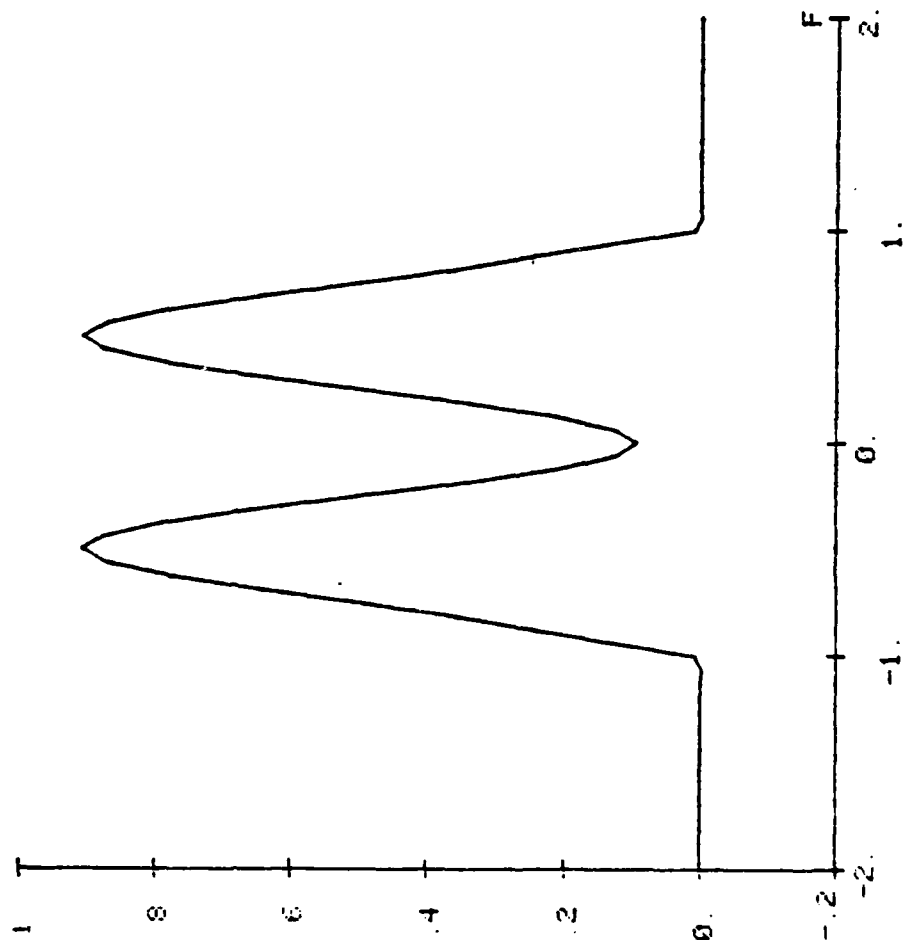
Figure 6.b Two-step procedure 2. A

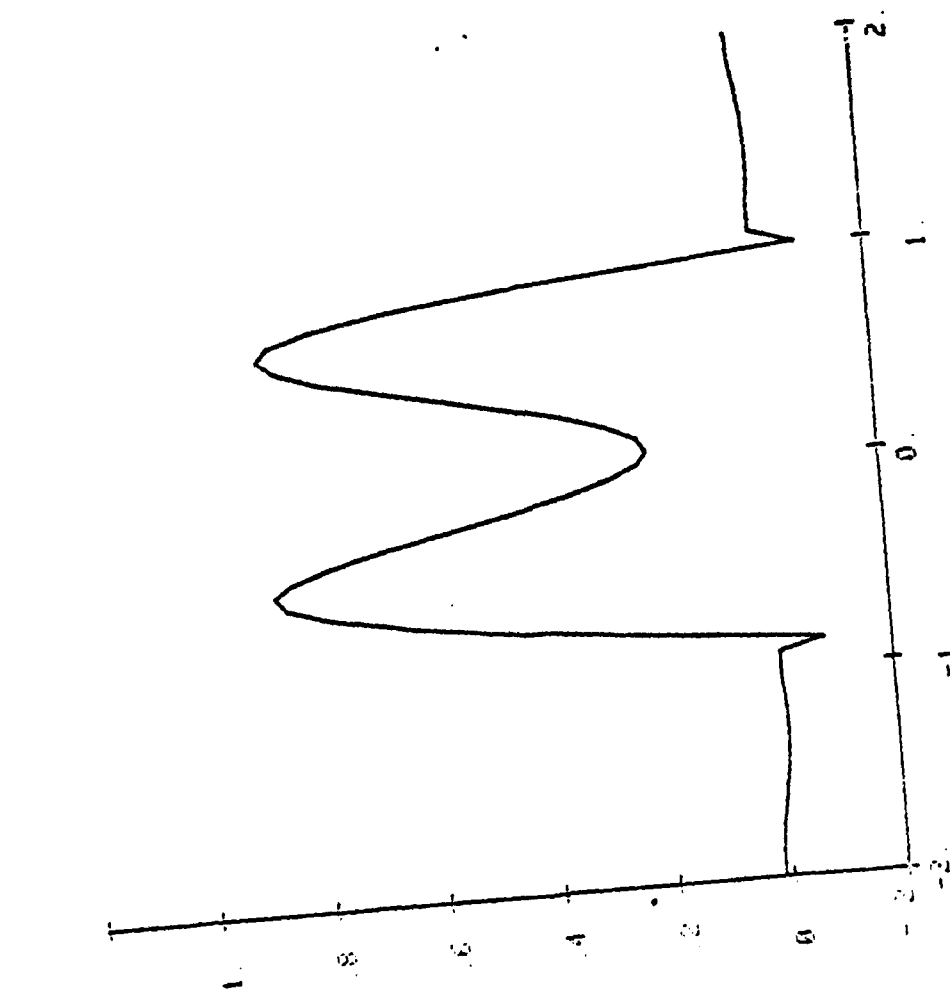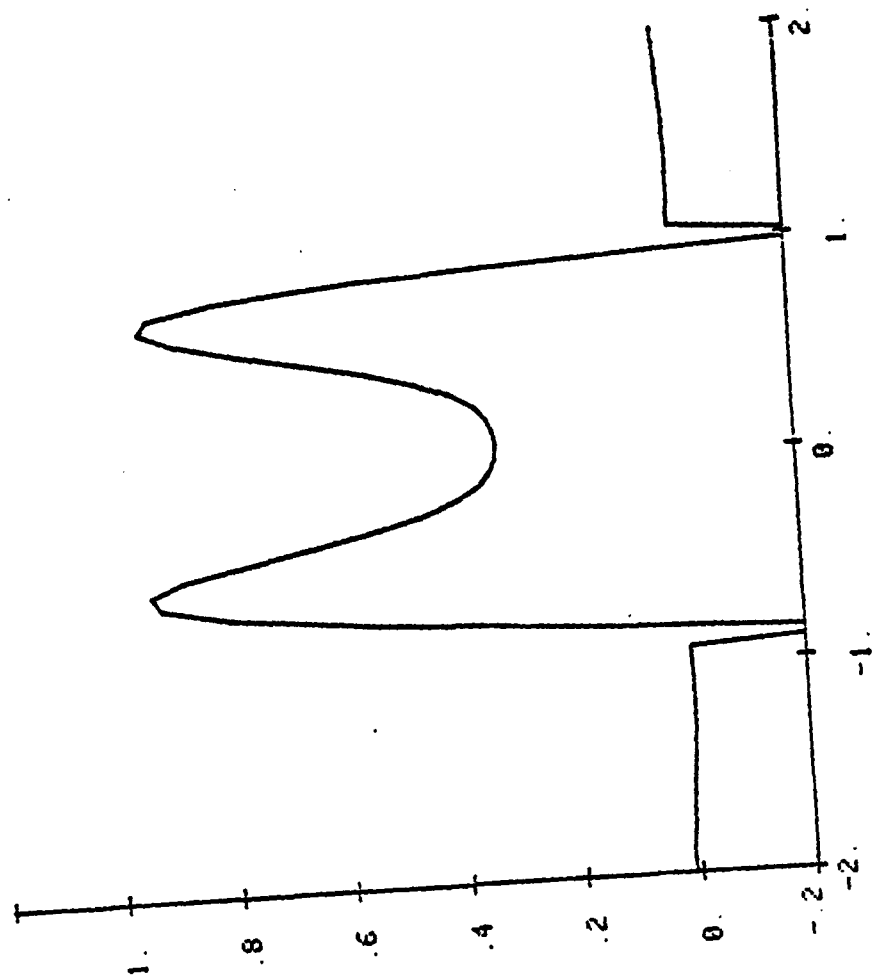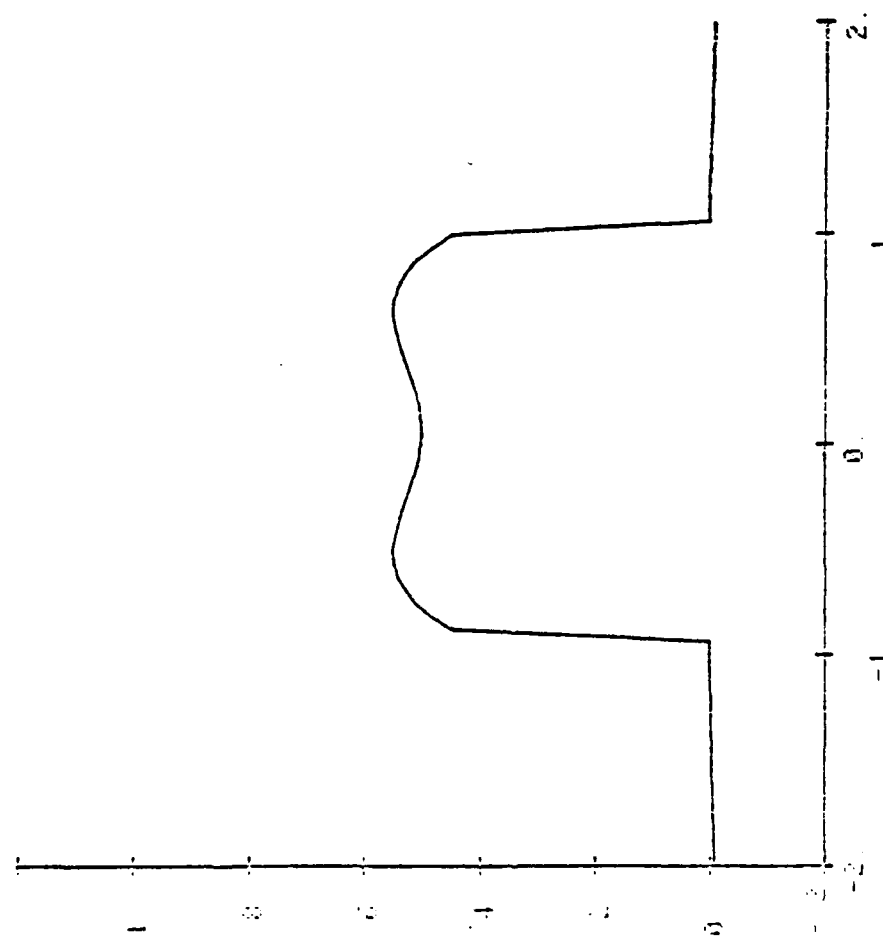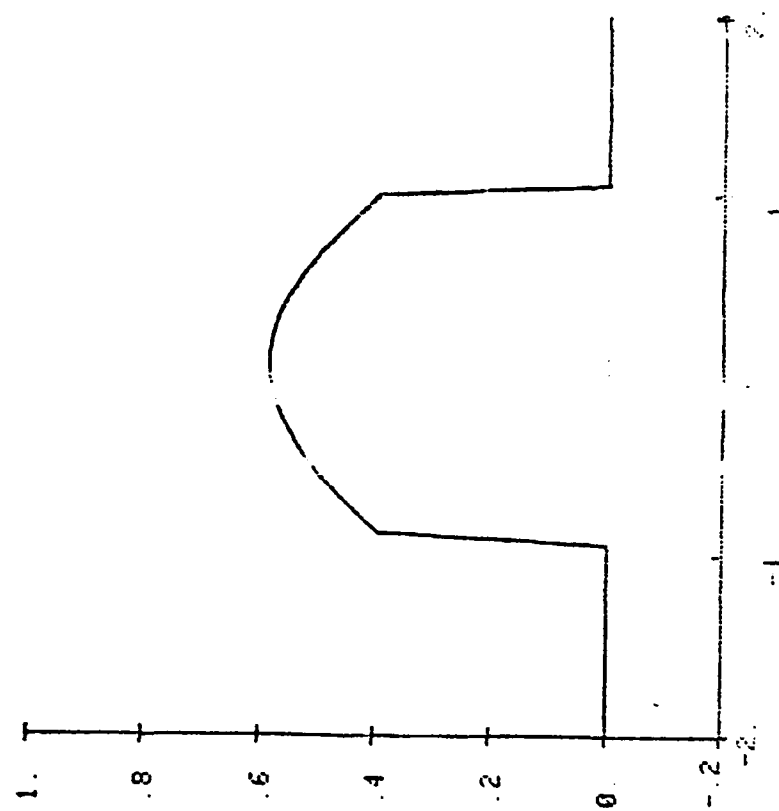Figure 7.b. 500 iterations. Second-digit noise. $\lambda = 1$.

Figure 7.a. 100 iterations. Second-digit noise. $\lambda = 1$.

Figure 7.d. 500 iterations. Second-digit noise.

Figure 7.c. 100 iterations. Second-digit noise.
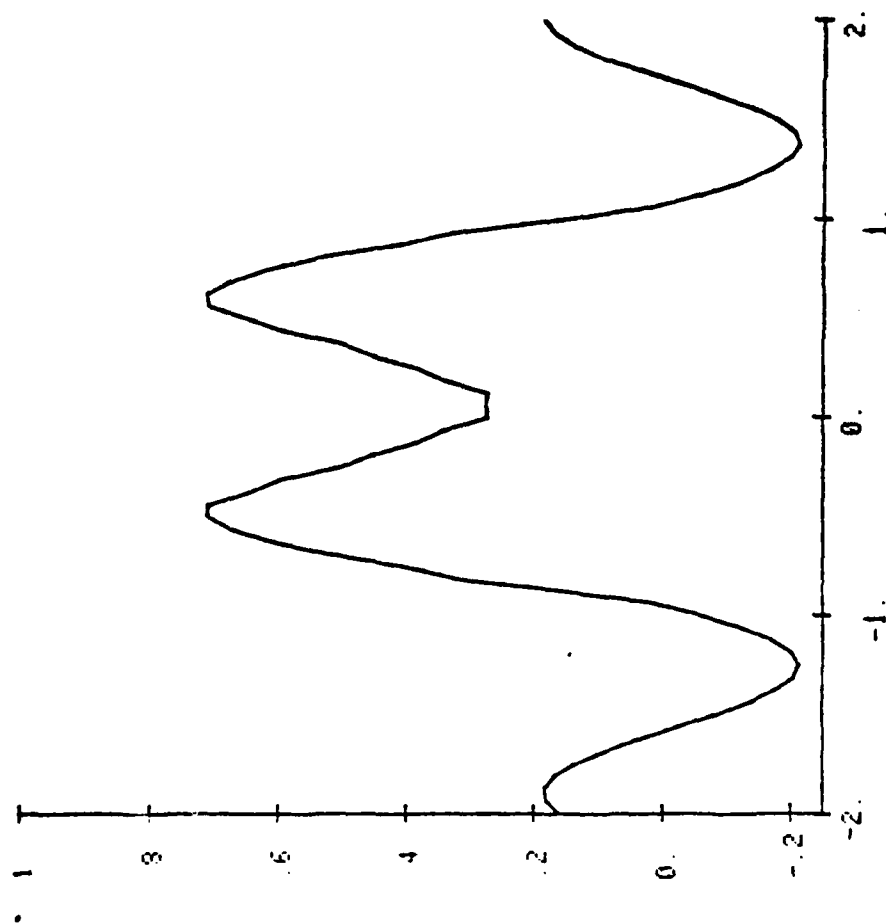
Figure 7f. 1000 iterations. $\lambda = \frac{1}{2}$. First digit noise.
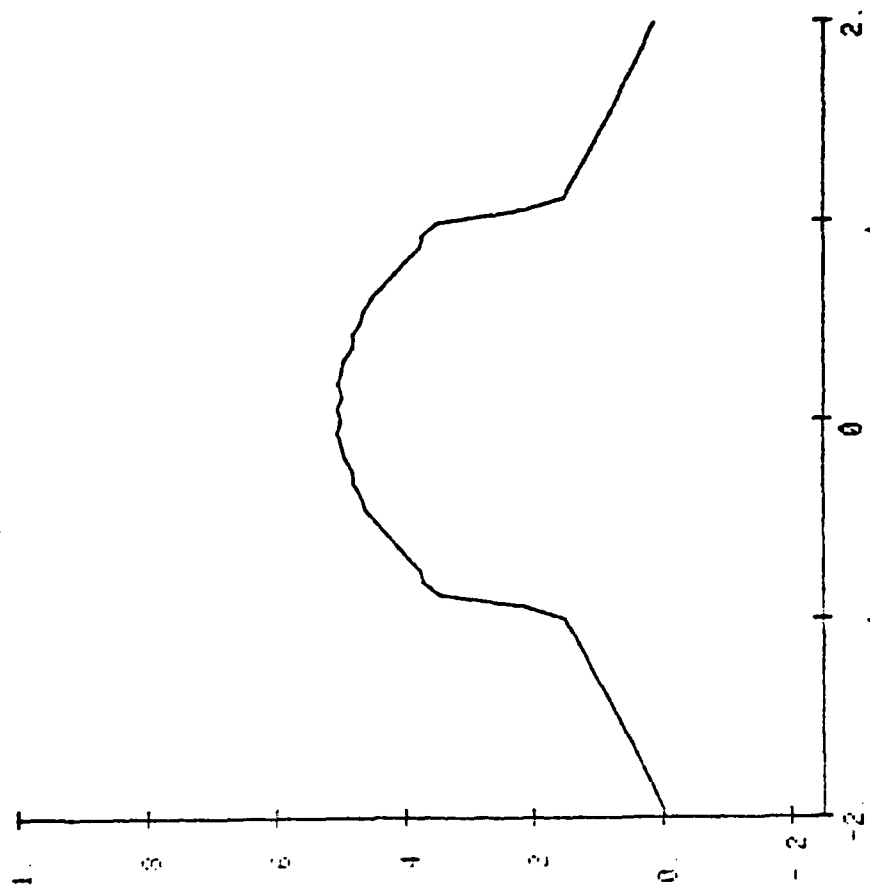
Figure 7e. 1000 iterations. $\lambda = 1$. First digit noise.

I.36

Figure 8.4. λ 0.00001. Third digit noise. λ 1.

Figure 8.5. 1000 iterations. λ 1. First digit noise.
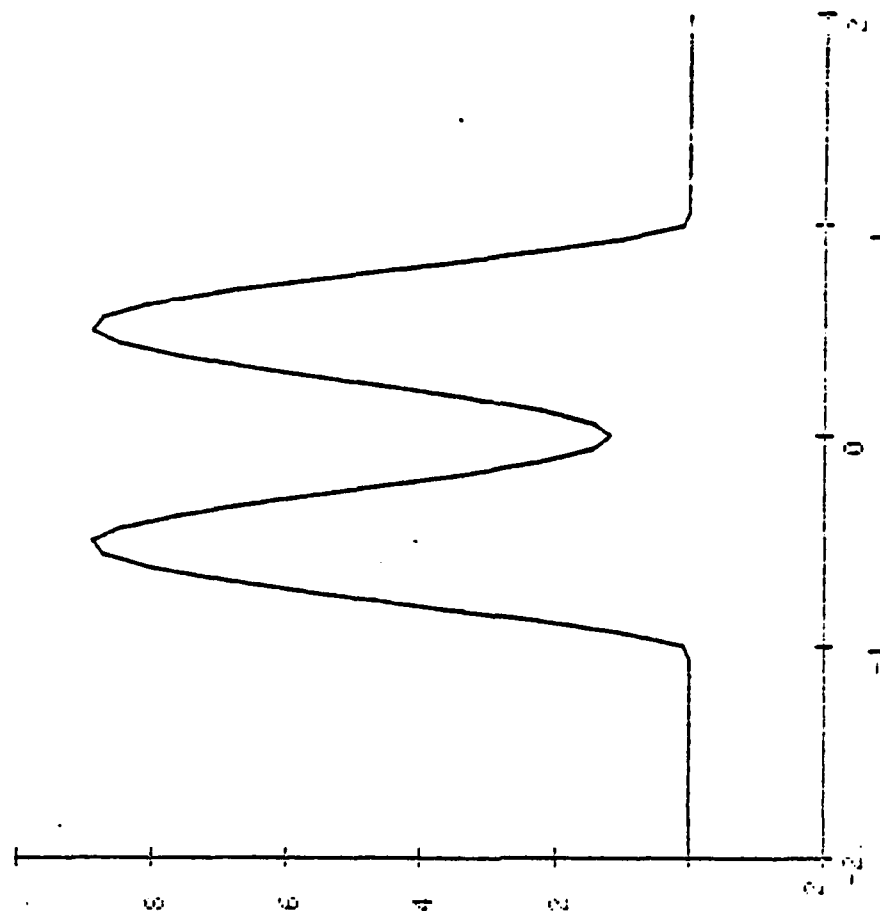
Figure 4.a. λ 0.001. Third-digit noise. A 1.
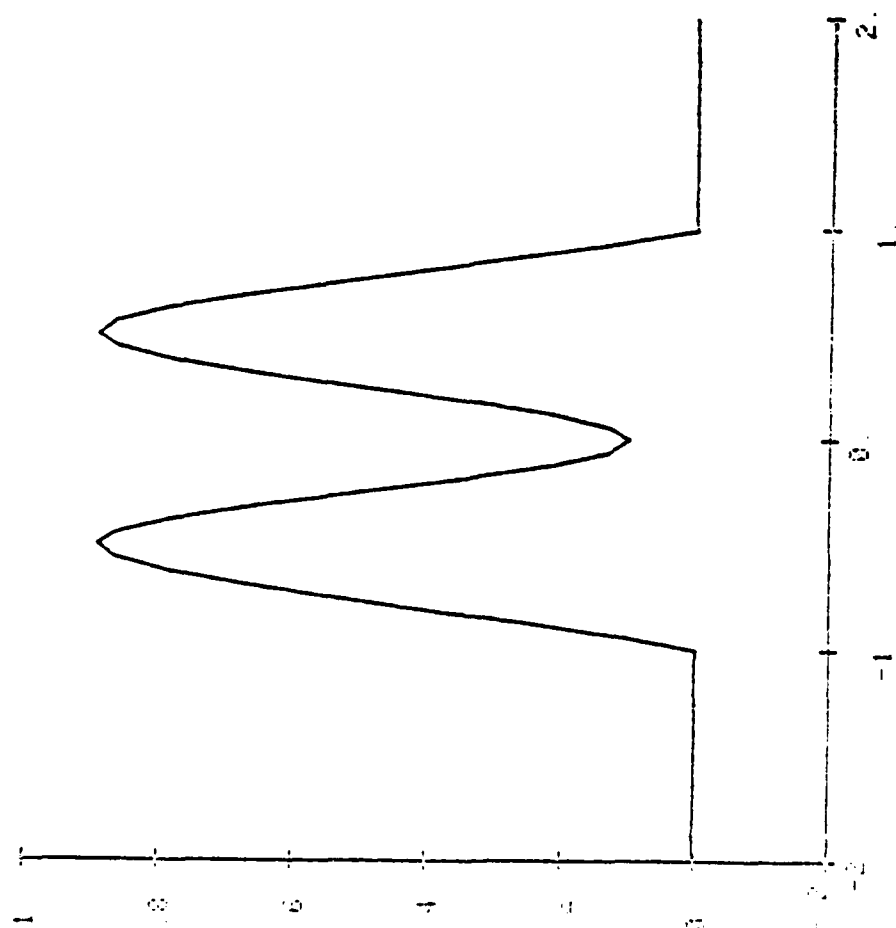
Figure 4.b. λ 0.001. Third-digit noise. A 1.
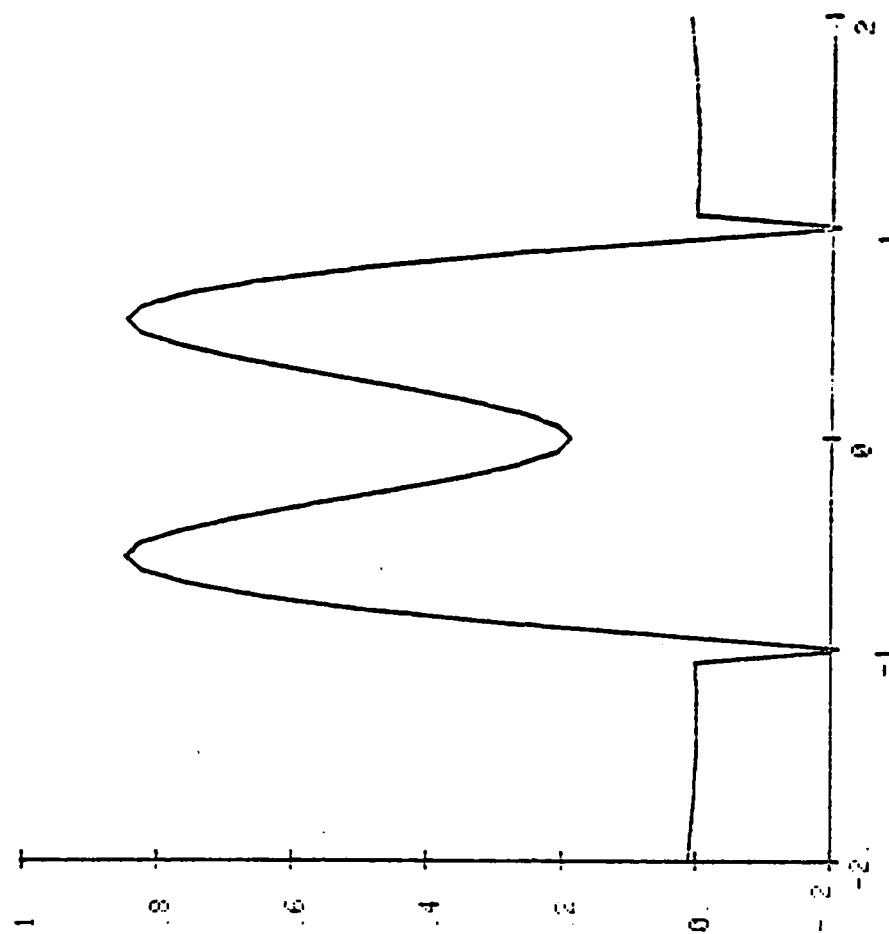
Figure 4.c. λ 0.001. Second-digit noise. A 1.
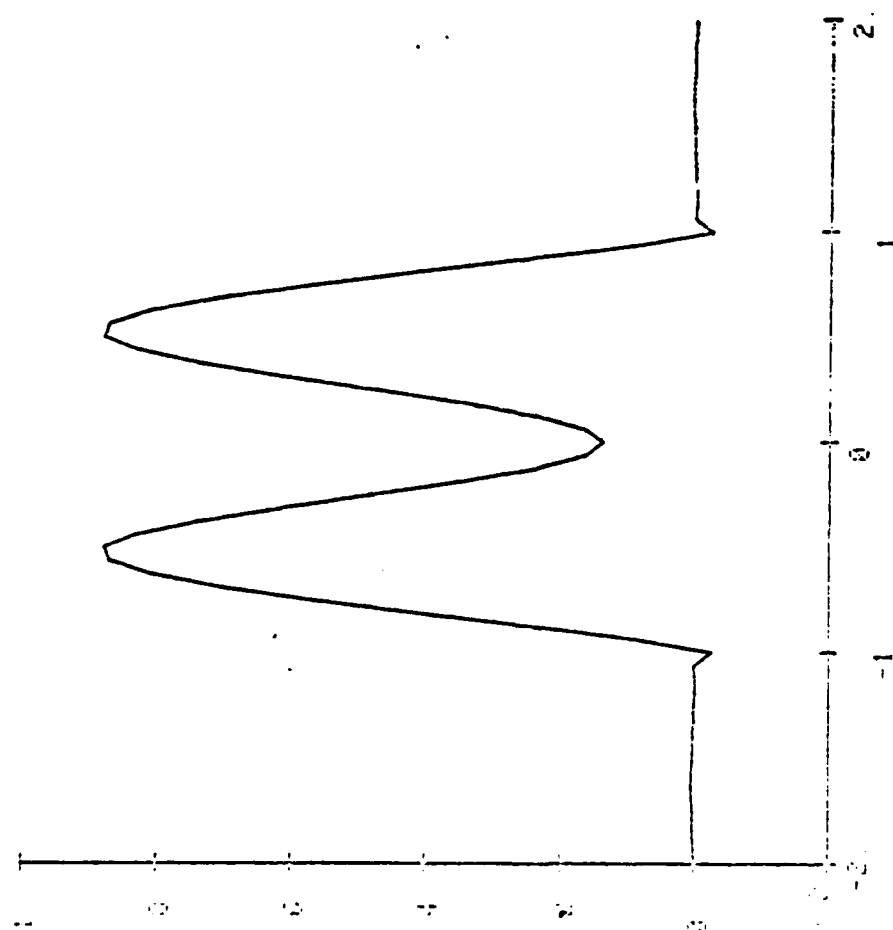
Figure 8.d. λ 0.001. Third-digit noise. A 1.
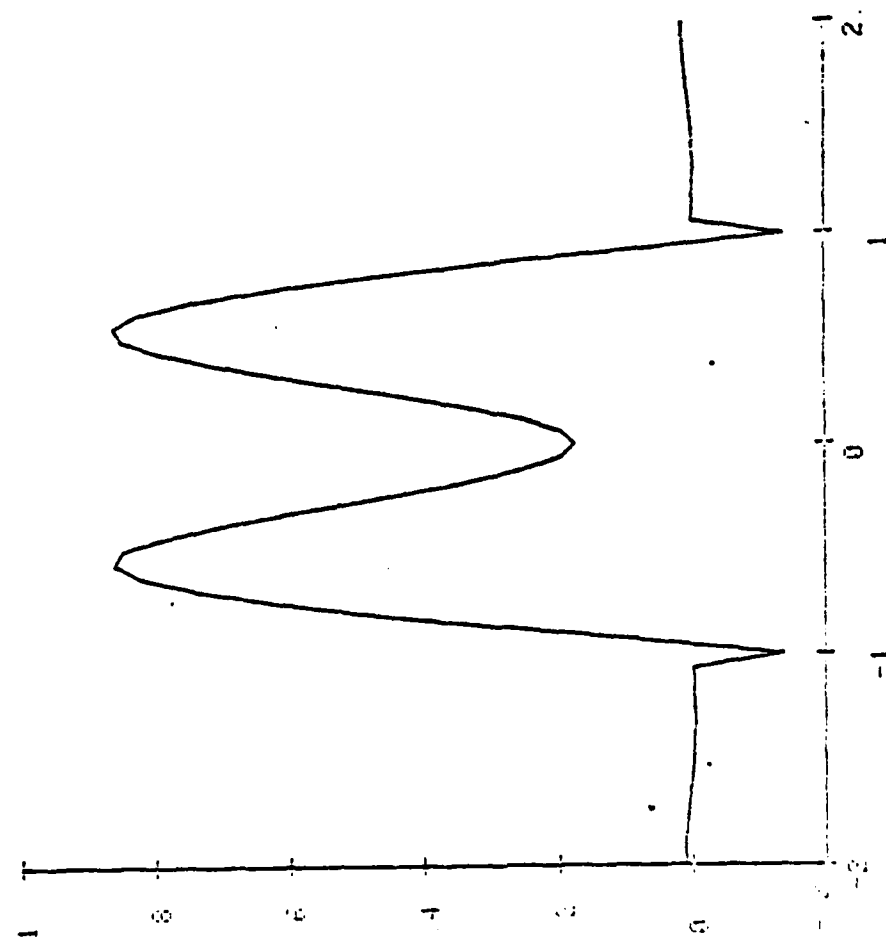
Figure 9.a. λ 0.1. Second-digit noise. $\times$ 1.

Figure 9.b. λ 0.01. Second-digit noise. $\times$ 1.

Figure 41. $\lambda = 0.5$. First digit noise. $N = 1$.

Figure 40. $\lambda = 0.5$. Third-digit noise. $N = 1$.

I.41

Figure 10.4. λ 0.00005. Second-digit noise. A 1.

Figure 10.3. λ 0.1. First-digit noise. A = 1.

Figure 10.c. λ 0.001. Second-digit noise. A

Figure 10.b. λ 0.0001. Second-digit noise. A

I.43

Figure 10c. $\gamma = 0.001$. First digit noise $\Lambda = \frac{1}{2}$.

Figure 10d. $\gamma = 0.01$. First digit noise. $\Lambda = \frac{1}{2}$.

I.44

Figure II.6. λ 0.00001. Second-digit noise. x

Figure II.6. λ 0.00001. Second-digit noise. x

Figure IId. λ = 0.00005. First-digit noise. A = 1.

Figure IIe. λ 0.00125. Second-digit noise. A 1.

I.46

Figure 11.6. $\lambda = 10^{-2}$. Third-digit noise. λ
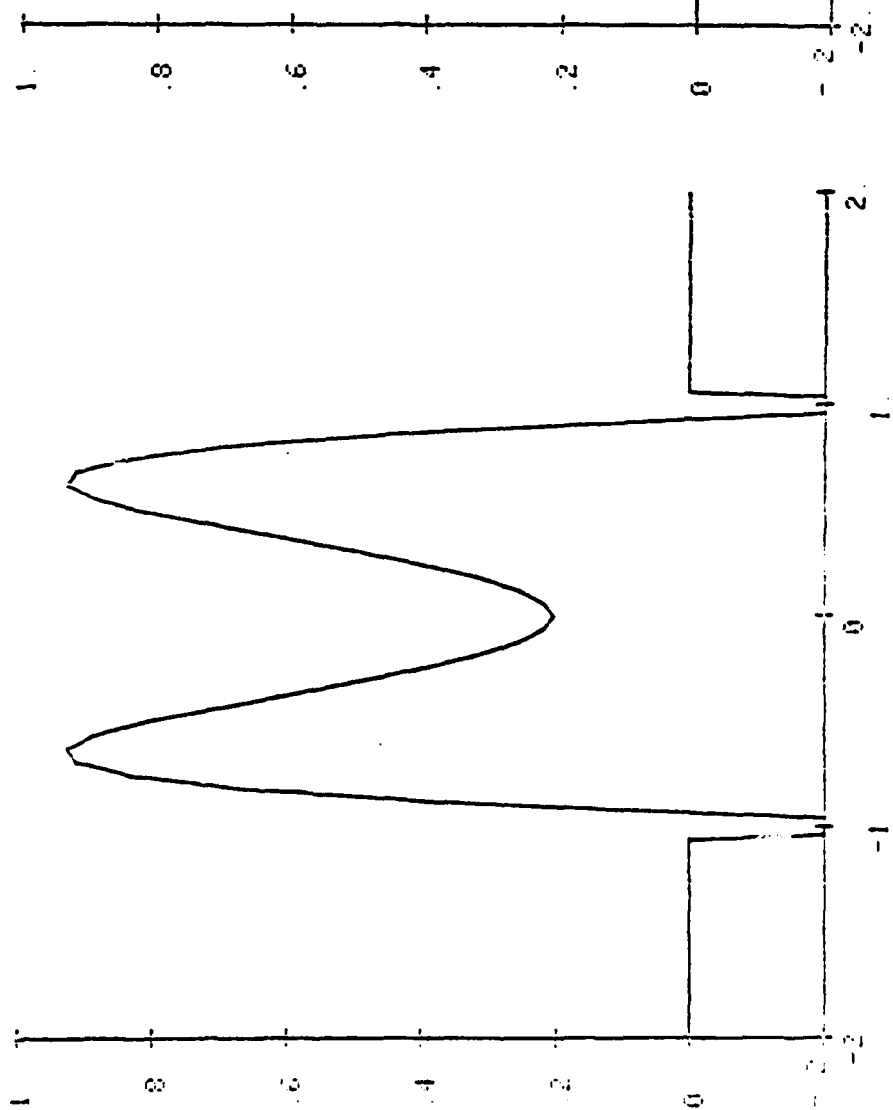
Figure 11.5. $\lambda = 0.00001$. First-digit noise. $\lambda = 1$.

I.47

Figure 12.a. $\lambda = 0.00125$. Third-digit noise. $A = 1$.

Figure 12.b. $\lambda = 10^{-6}$. Third-digit noise. $A = 1$.

Figure 11.4. $\times$ 10^{-2}. Third-digit noise. $\lambda = 1.$

Figure 12.4. $\lambda = 0.001$. Third-digit noise. $\lambda = 0.$
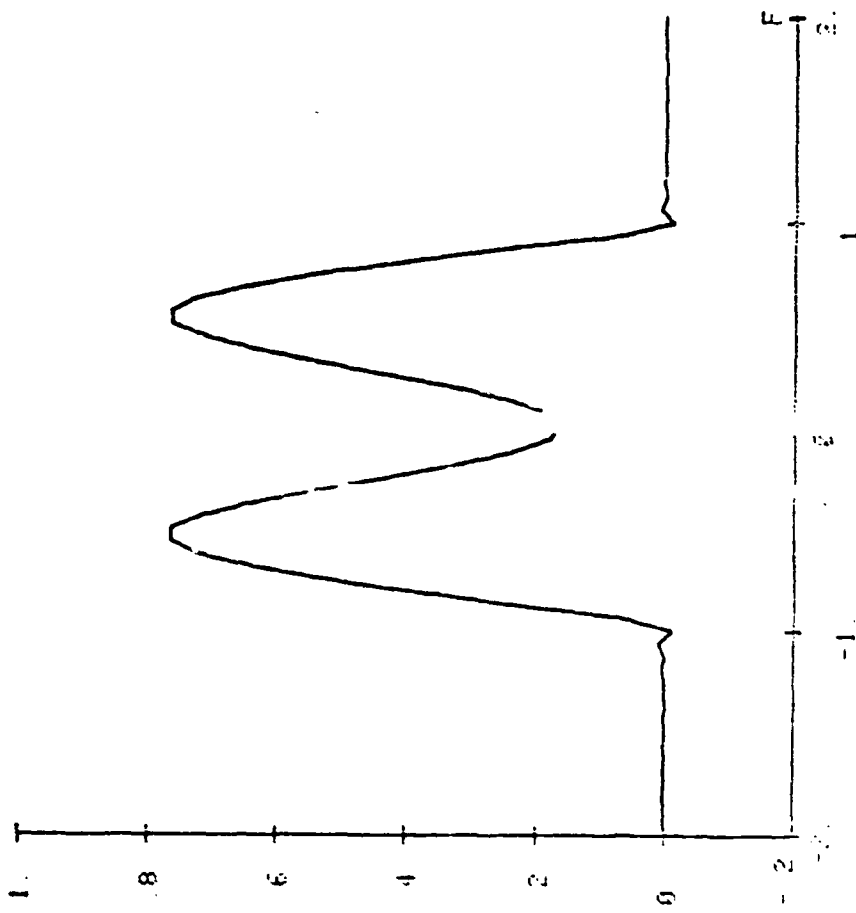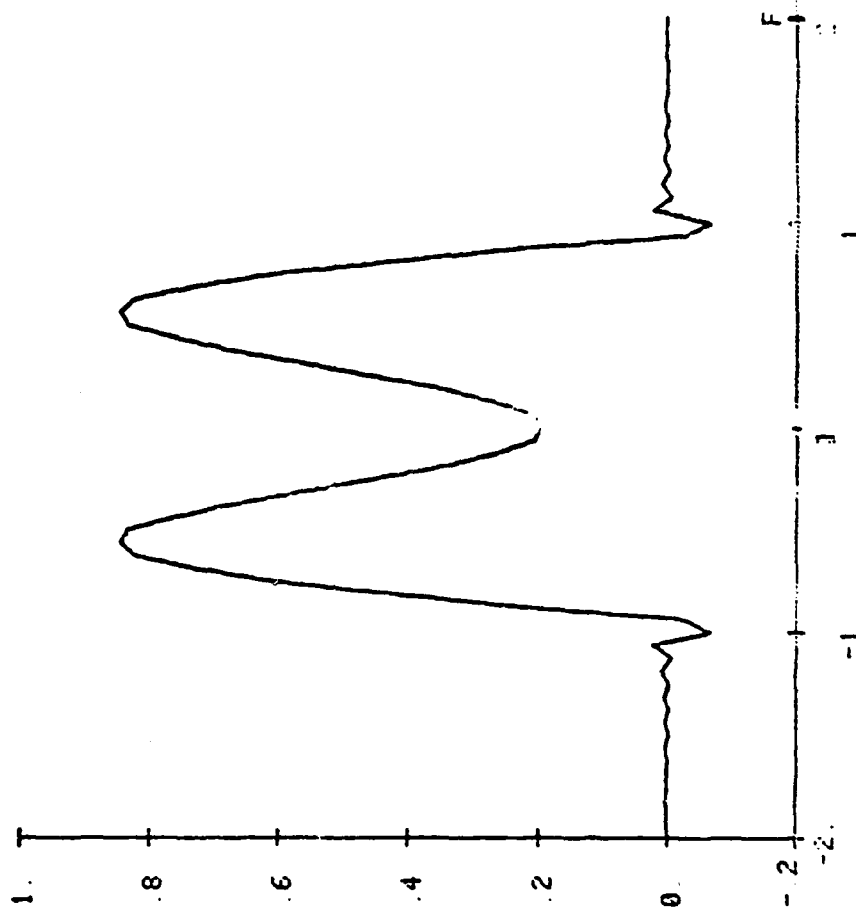
Figure 11.b. 10^{-2}. Second-digit noise. A 1.

Figure 11.a. λ 10^{-1}. Second-digit noise. A 1.

Figure 13c. $\gamma = 10^{-0.5}$. First digit noise. $A = 1$.
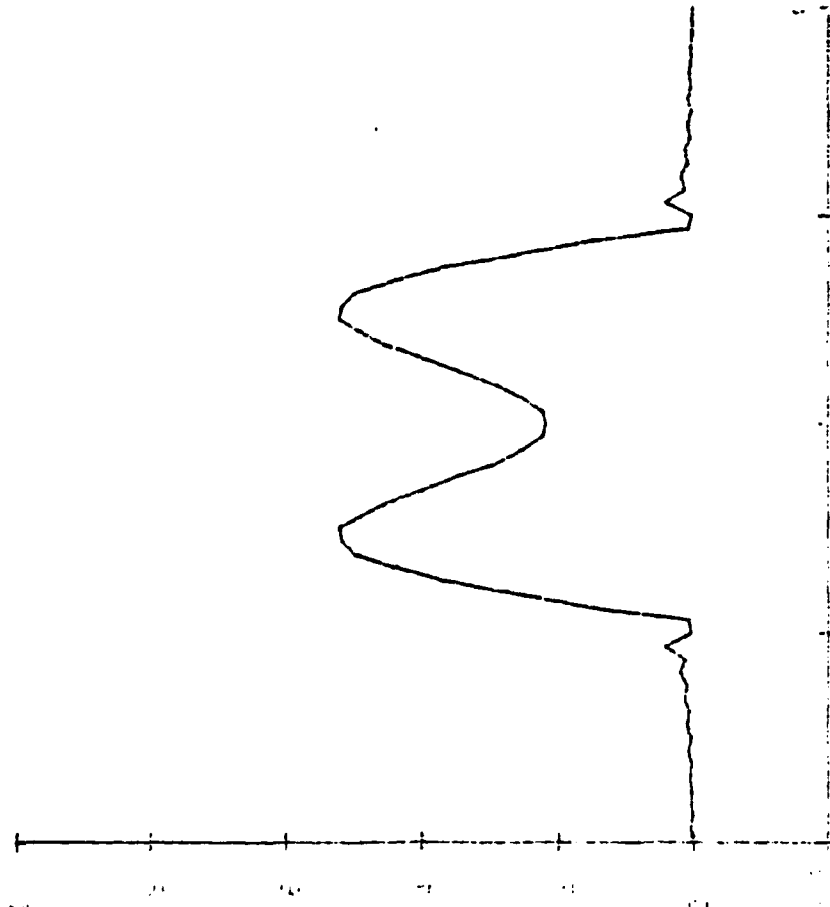
Figure 13d. $\gamma = 0.1$. First digit noise. $A = 1$.
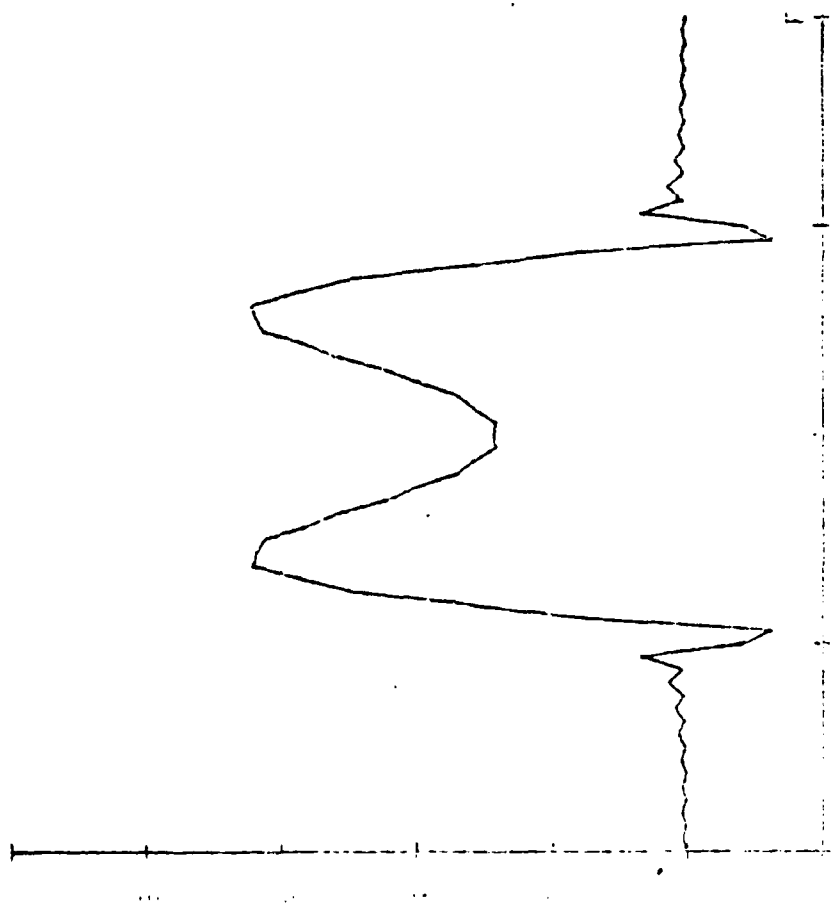
Figure 14.b. $\lambda \sim 10^{-3.5}$. Third-digit noise. $A = \frac{1}{2}$.
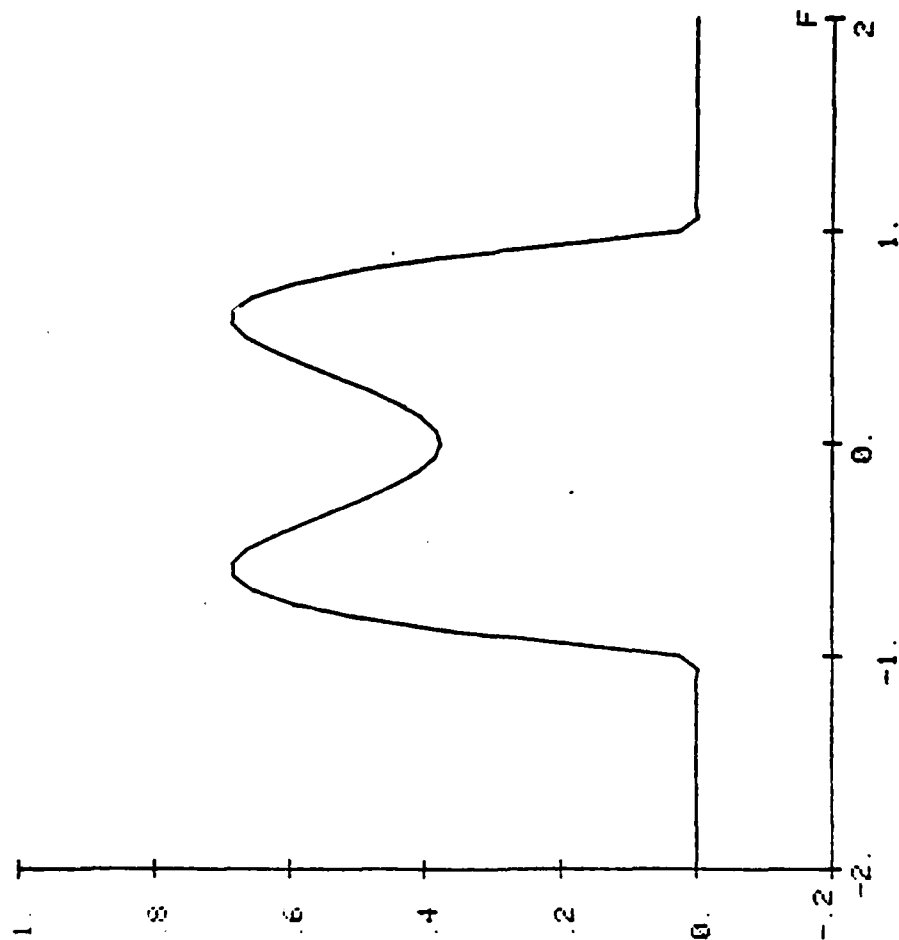
Figure 14.a. $\lambda \sim 10^{-3}$. Third-digit noise. $A = \frac{1}{2}$.

Figure 15... λ 10^{-3}. Second-digit noise. A $\mathfrak{d}_2$.
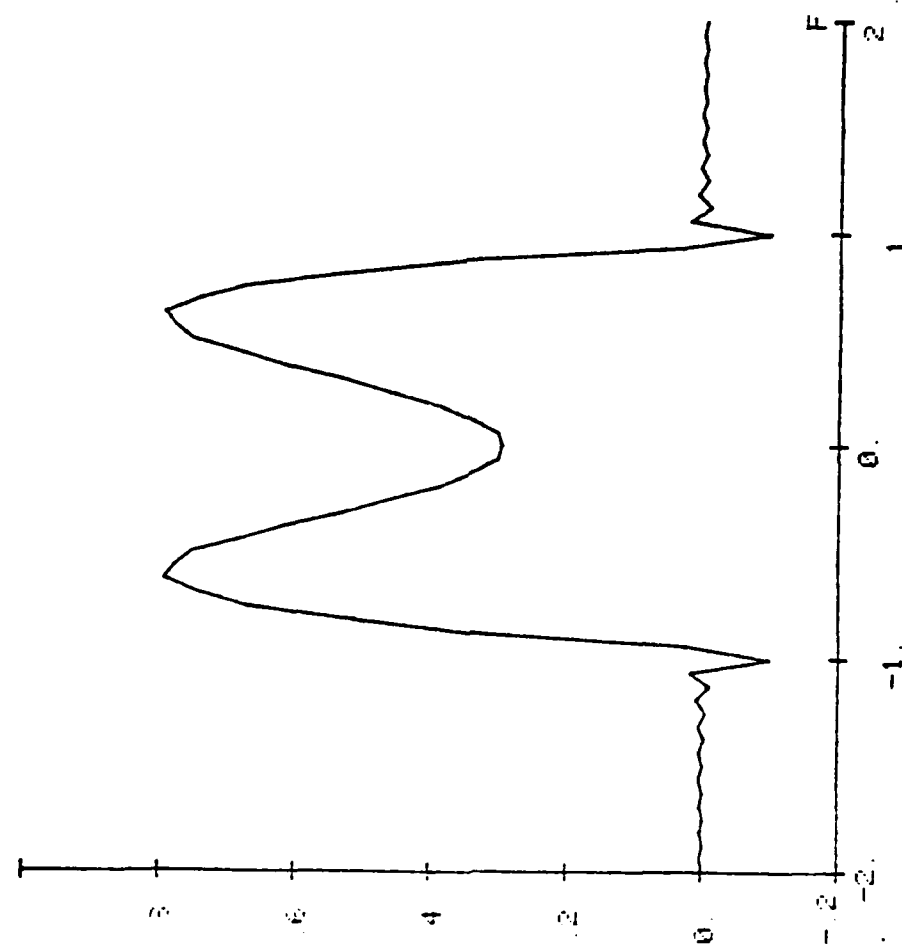
Figure 15... λ 10^{-4}. Third-digit noise. A $\mathfrak{d}_2$.

Figure 15c. $\lambda = 10^{-2.8}$. First digit noise. $A = \frac{1}{2}$.
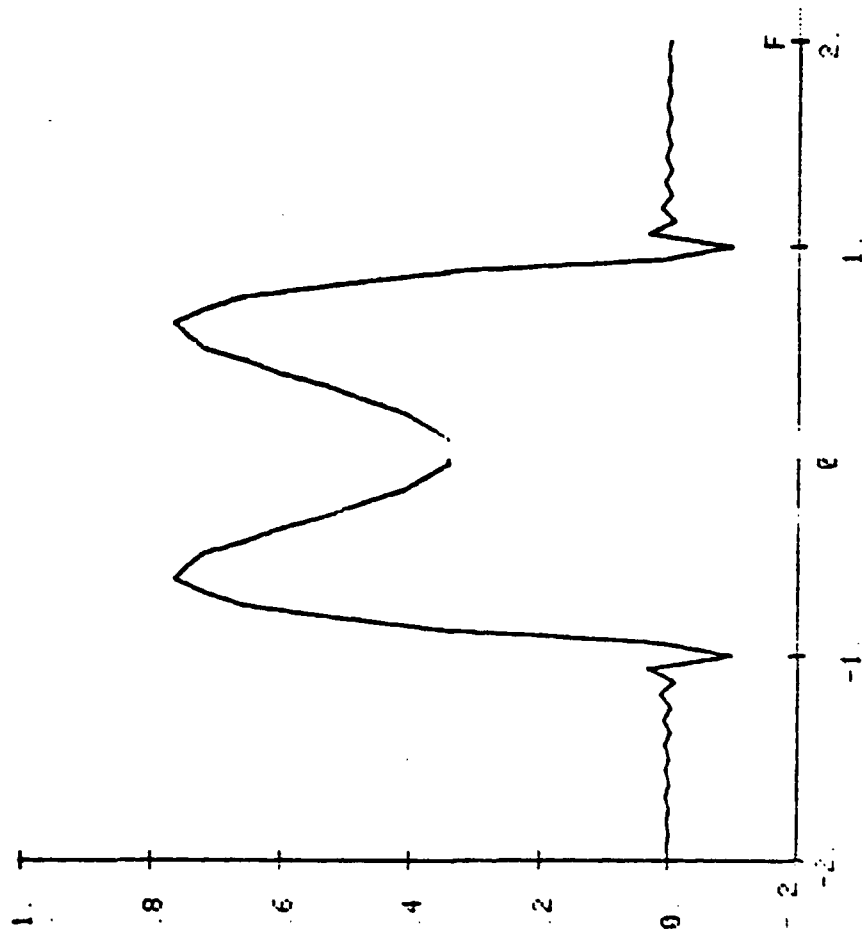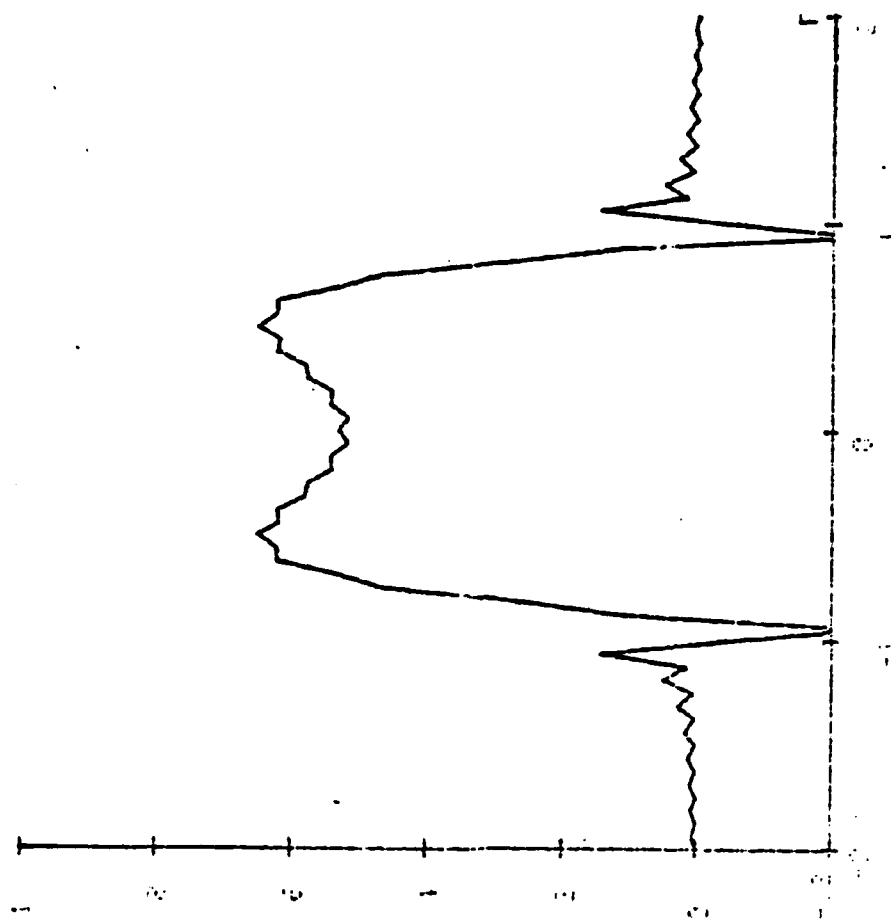
Figure 15.b. $\lambda = 10^{-1.5}$. Second-digit noise. $A = \frac{1}{2}$.

Figure 16.a. λ 10^{-6}. Third-digit noise. A
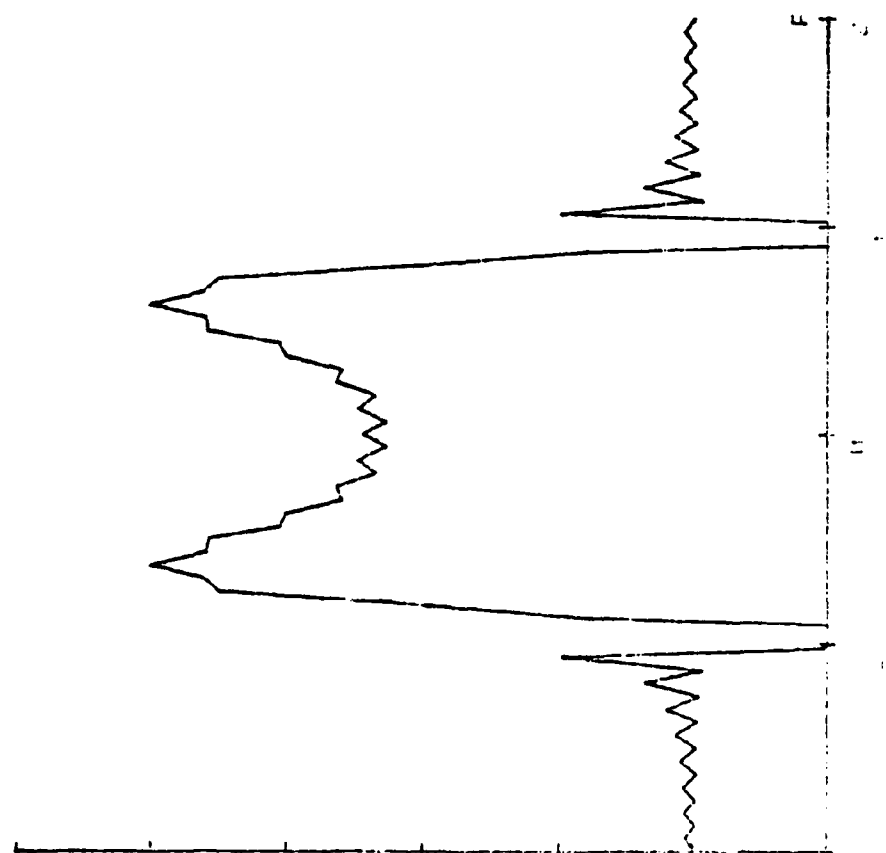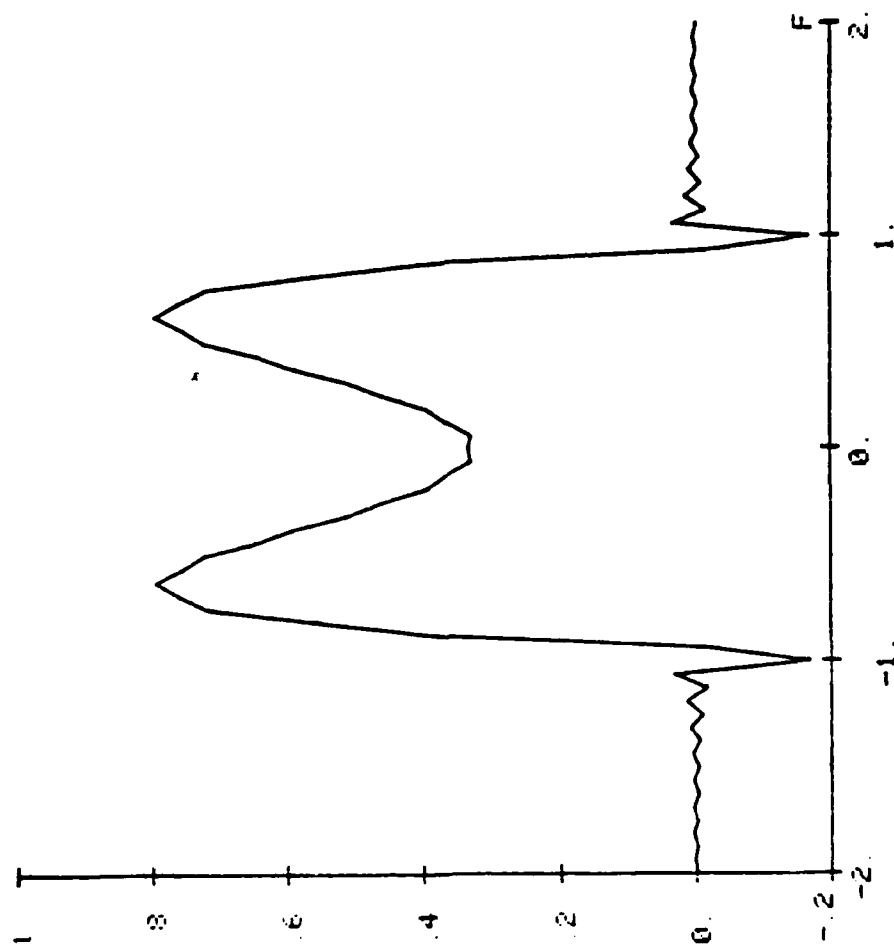
Figure 15d. λ 10^{-14}. First-digit noise. A
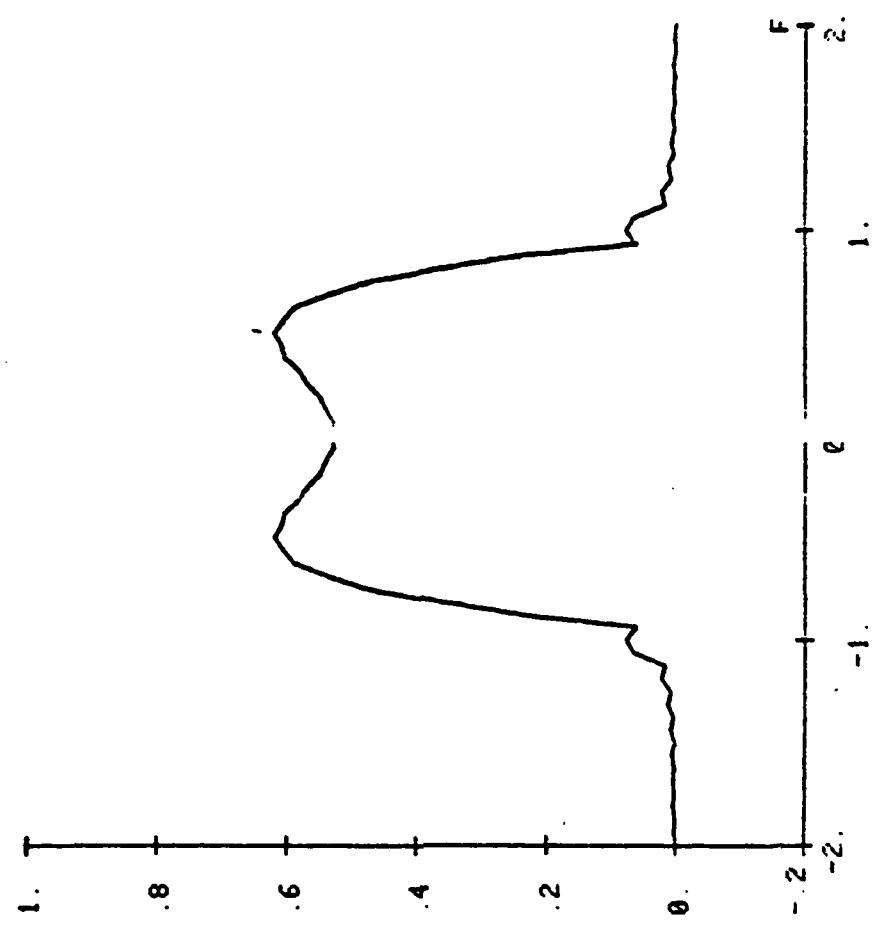
Figure 17.a. λ 10^{-5}. Second-digit noise. $A = 4$.

Figure 16.b. λ 10^{-5}. Third-digit noise. $A = 4$.
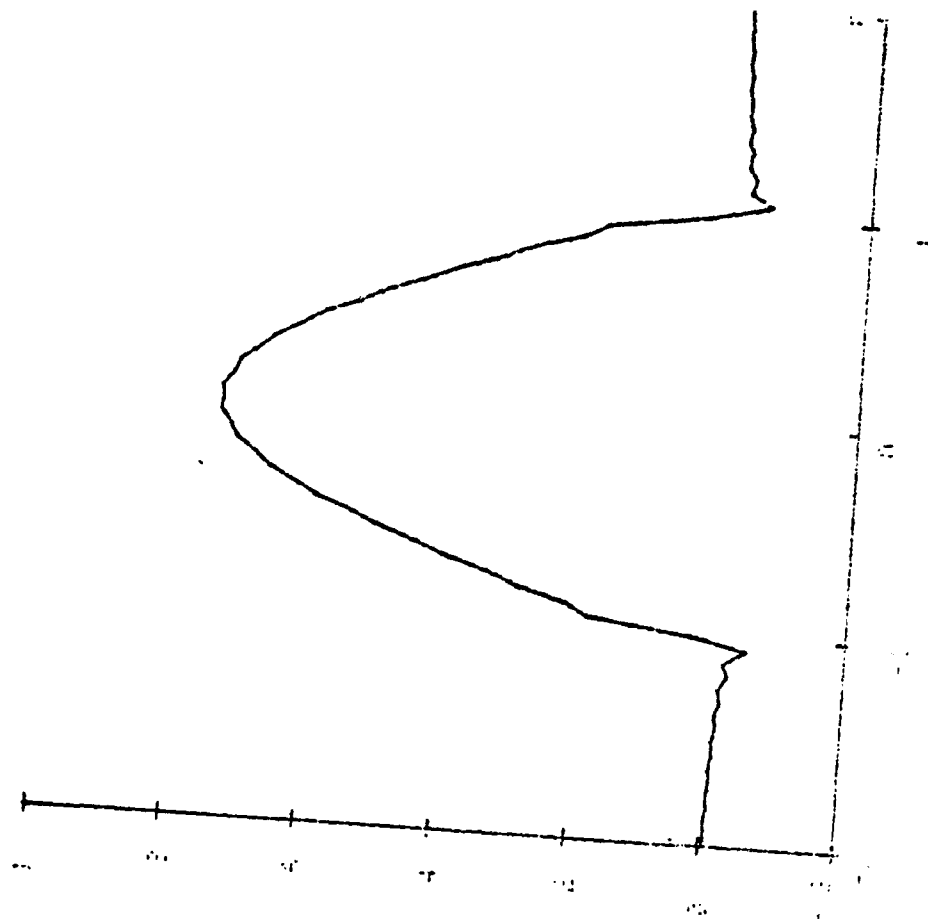
Figure 17c. $\gamma = 10^{-2}$. First-digit noise. A = 1.
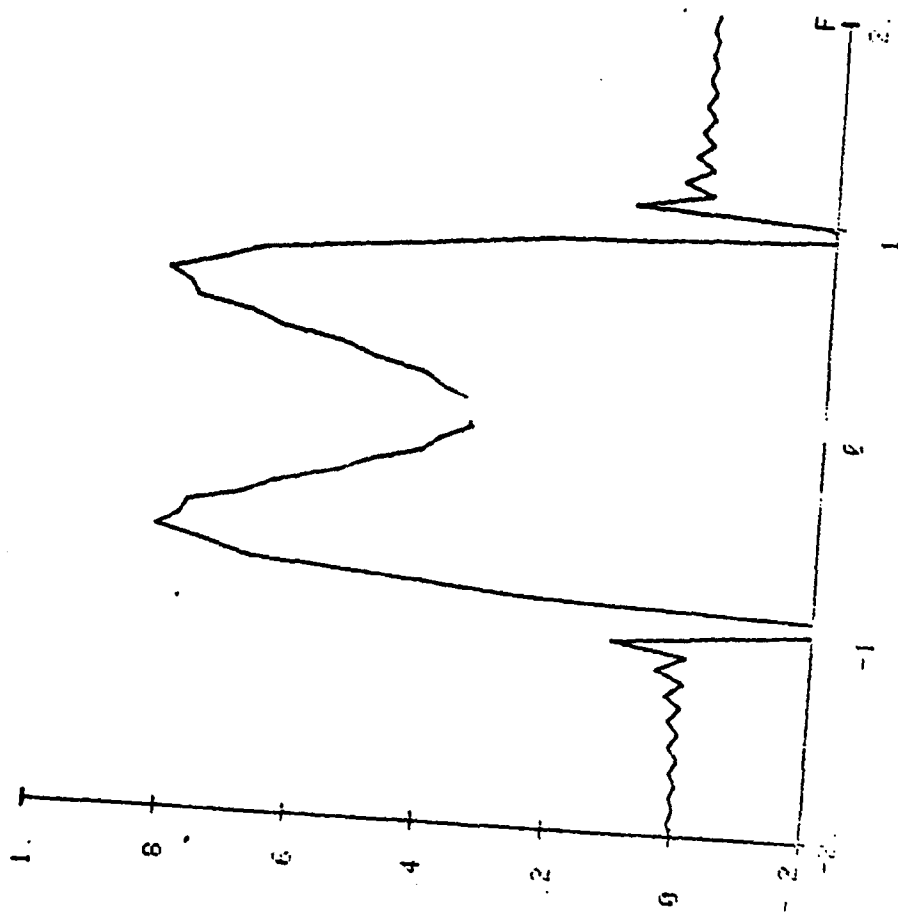
Figure 17b. $\gamma = 10^{-3.5}$. Second-digit noise. A = 1.

I.58

Figure 17e. $\lambda = 10^{-3.5}$. First digit noise. $A = \frac{1}{4}$.
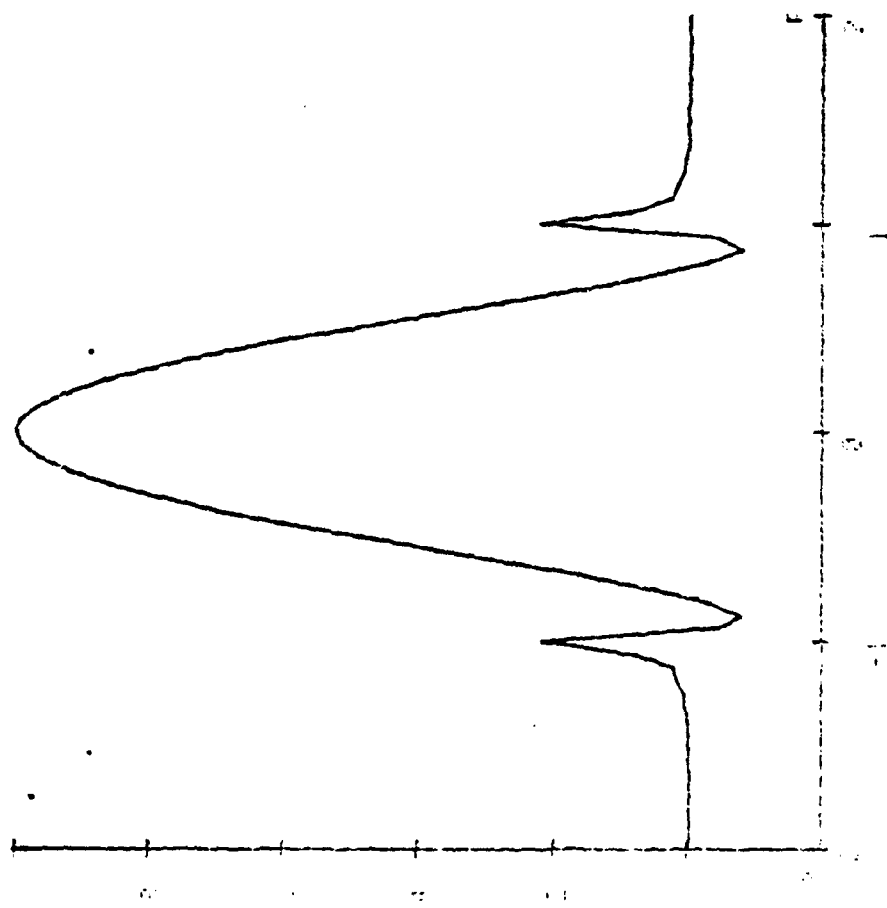
Figure 17d. $\lambda = 10^{-3}$. First digit noise. $A = \frac{1}{4}$.

Figure 18.b.   0. 200 iterations.   A   ½.
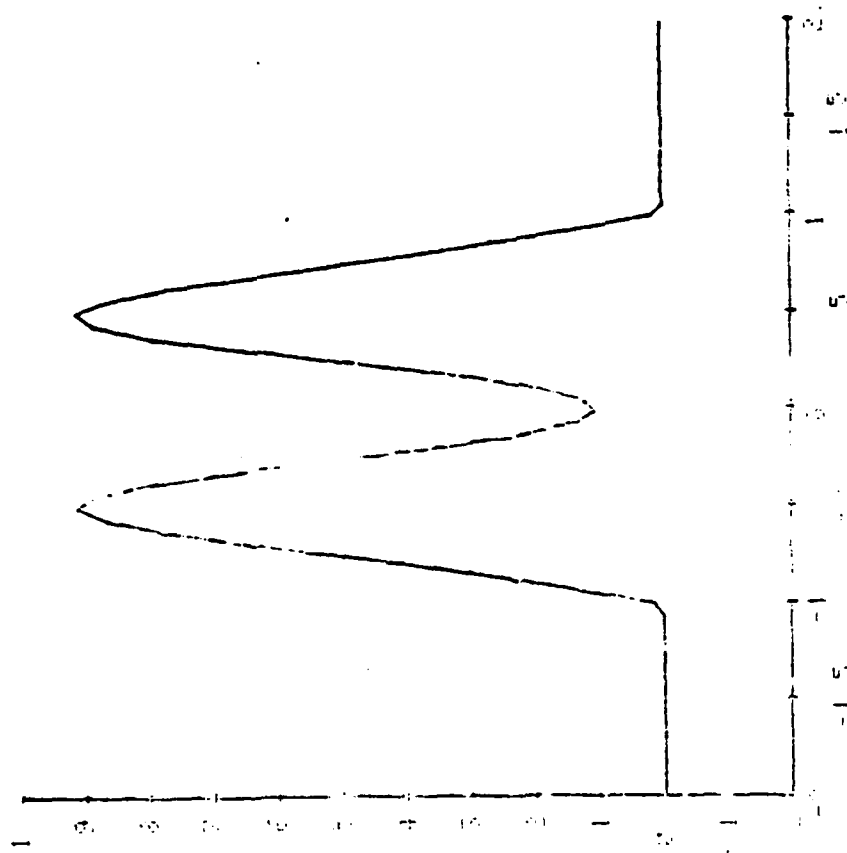
Figure 18.a.   0. 200 iterations.   A   1.
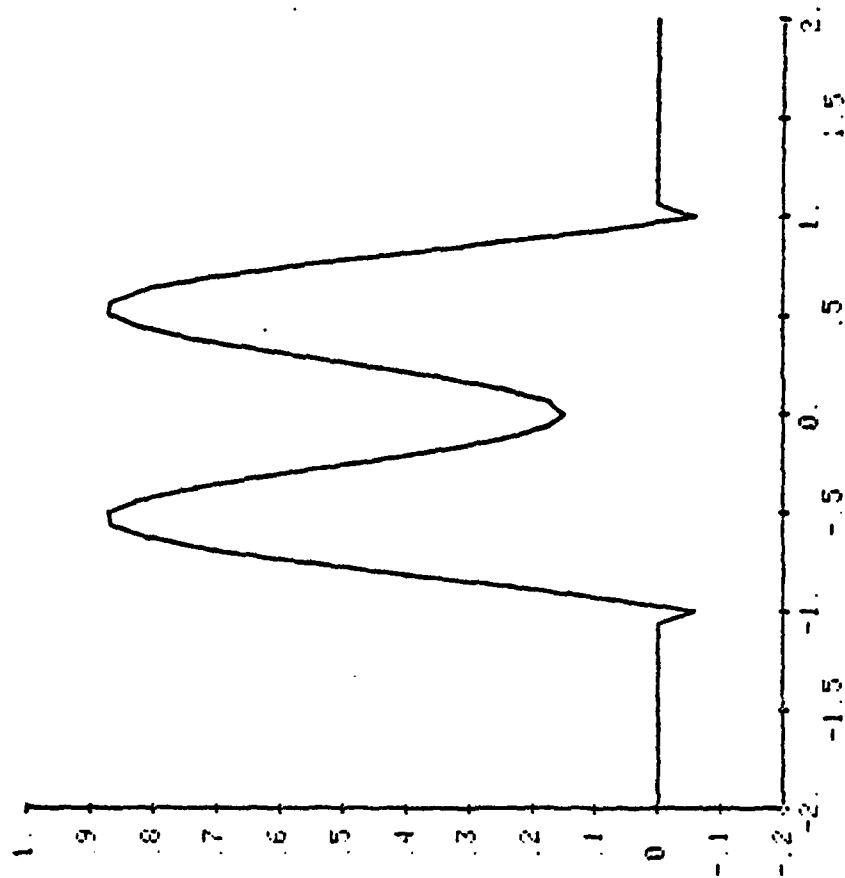
Figure 19.a. $\mu = 0$, 5 iterations. $A = 1$. Second-digit noise.
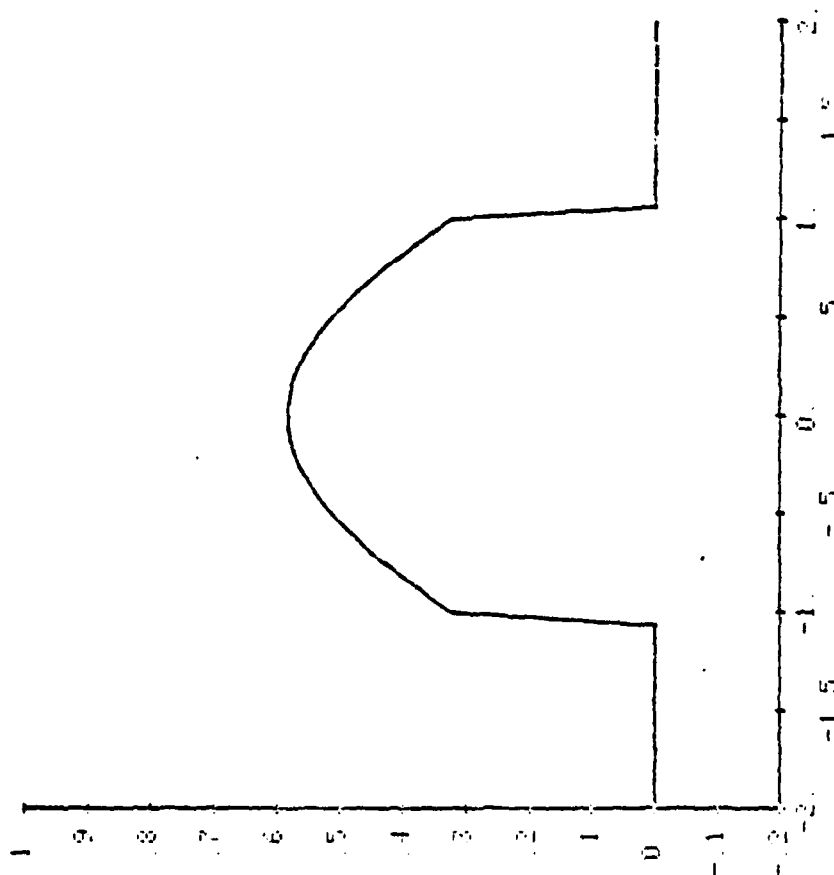
Figure 18.c. $\mu = 0$, 200 iterations. $A = 1$.
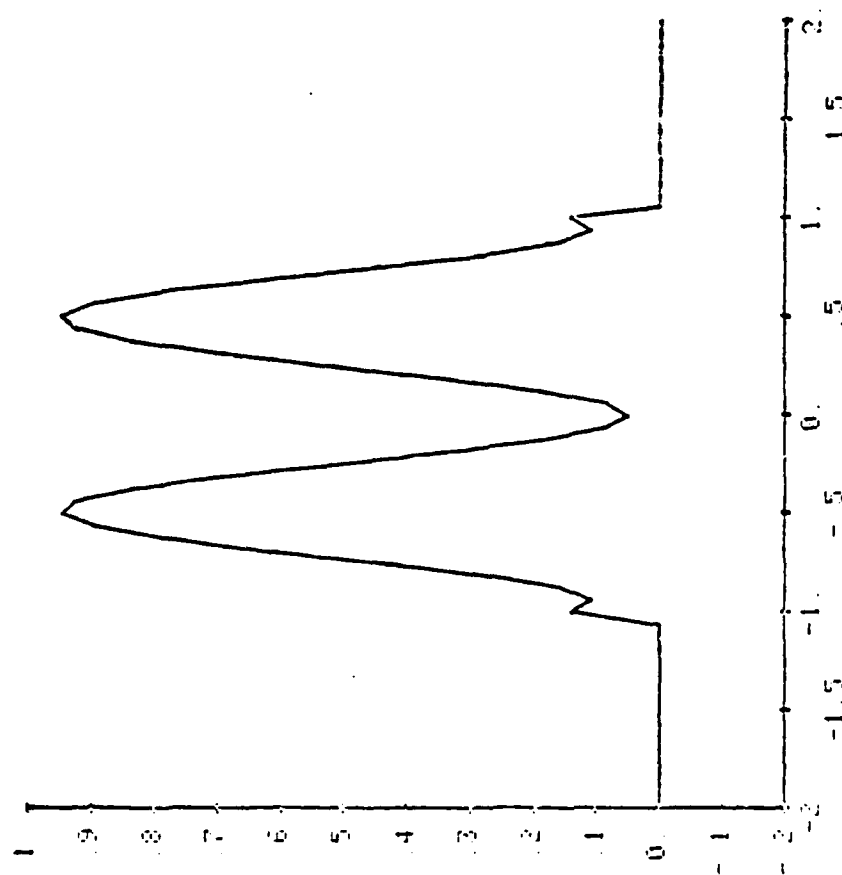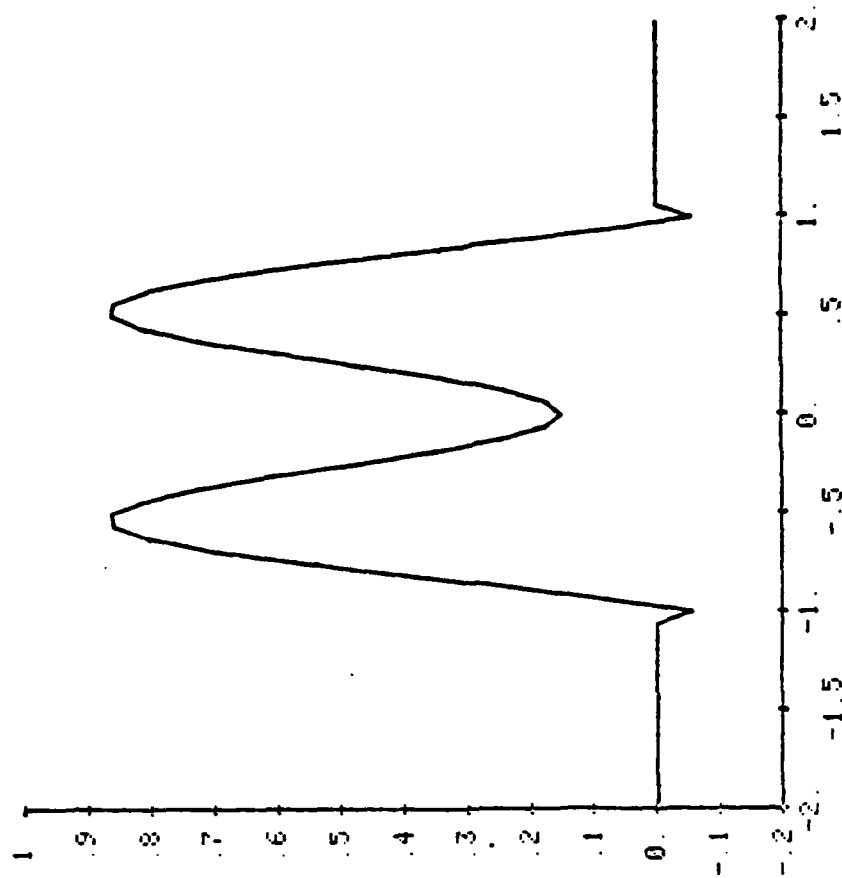
Figure 20.a. μ 0.01. 5 iterations. A 1. Second-digit noise.
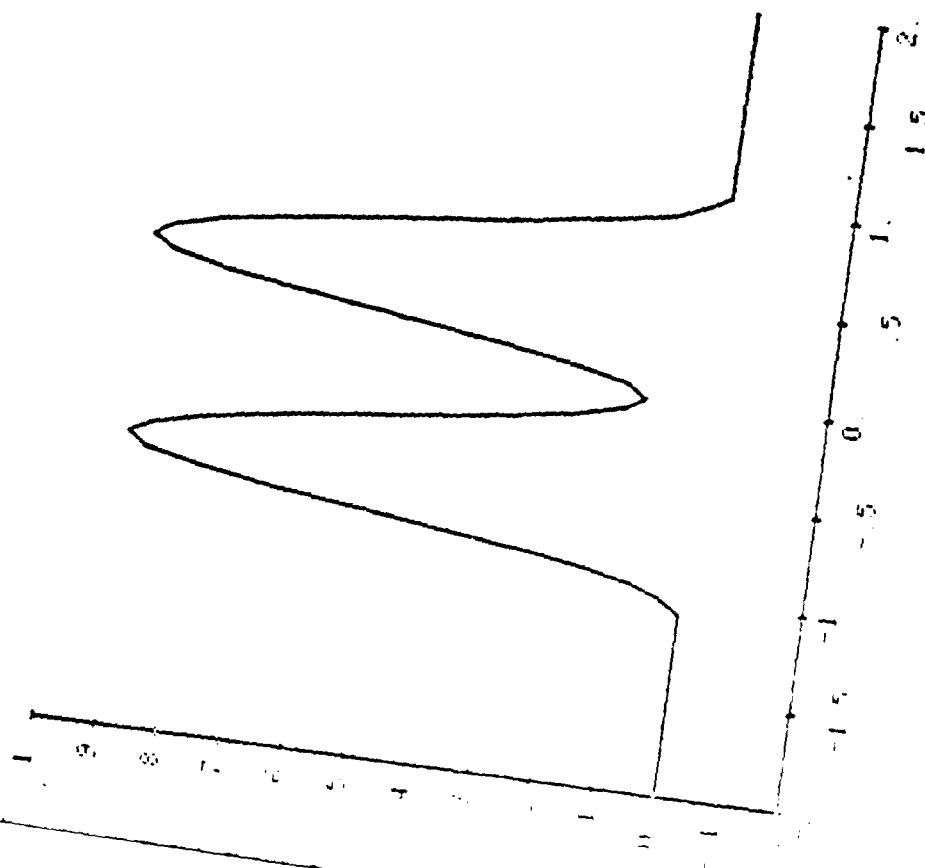
Figure 19.b. μ 0. 200 iterations. A 1. Second-digit noise.

Figure 21.a. μ = 0. 300 iterations. A = ½. Second-digit noise.
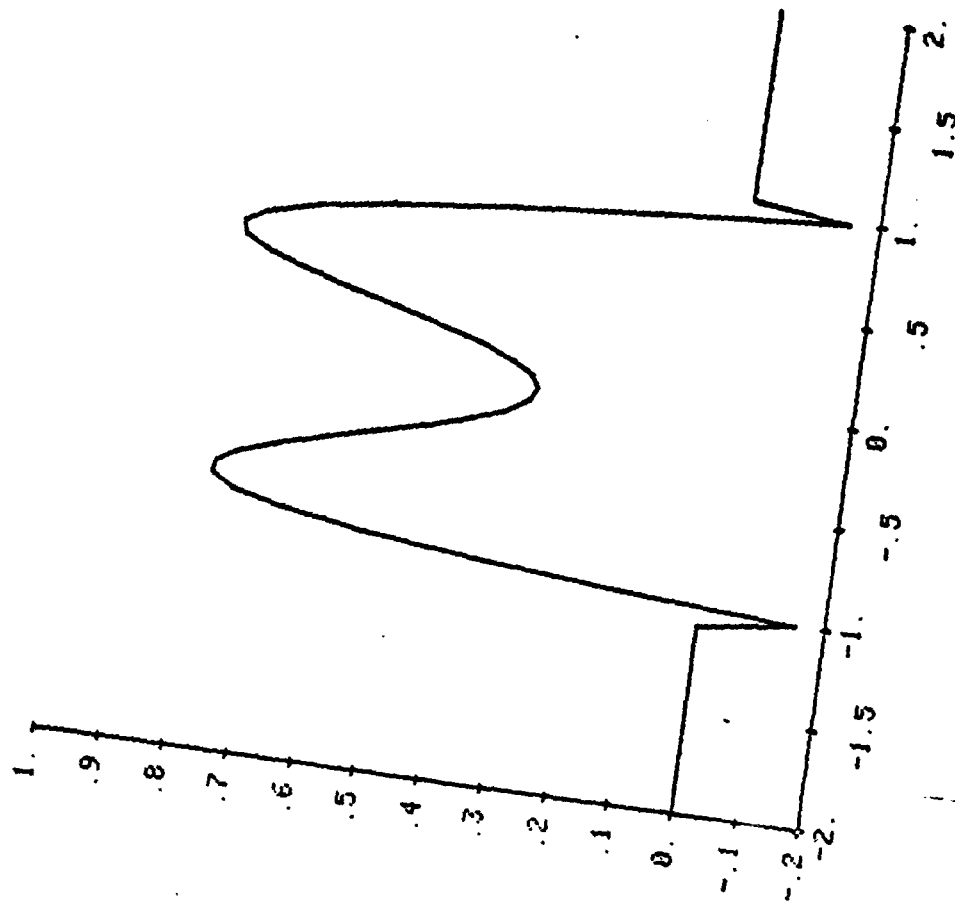
Figure 21.b. μ = 0.03. 200 iterations. A = 1. Second-digit noise.
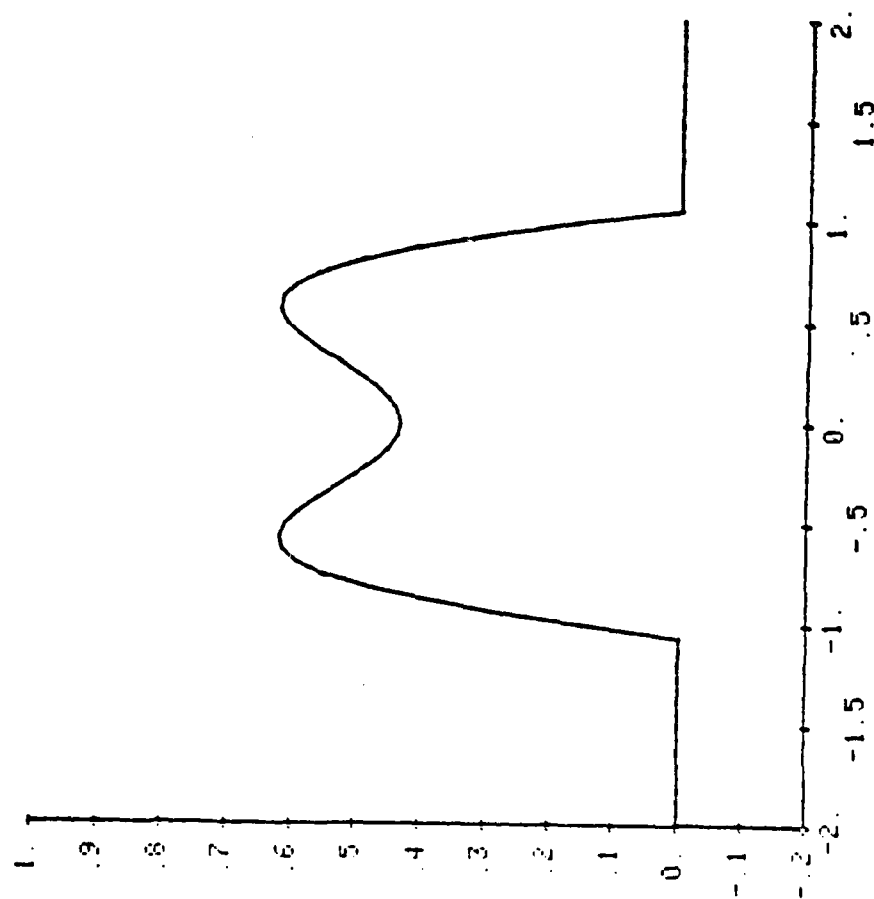
Figure 21.b. u = 0.005. 500 iterations. A = ½. Second-digit noise.

A UNIFIED HILBERT SPACE APPROACH TO ITERATIVE
LINEAR SIGNAL RESTORATION

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ABSTRACT

This paper deals with iterative solutions of the linear signal restoration problem: g=Af. First, several existing techniques for solving this problem with different underlying models are unified. Specifically, the following are shown to be special cases of a general iterative procedure (Bialy 1959 [1]) for solving linear operator equations in Hilbert spaces: 1. A Van Citter-type algorithm for deconvolution of discrete and continuous signals. 2. An iterative procedure for regularization when g is contaminated with noise. 3. Papoulis-Gerchberg's algorithm for extrapolation of continuous signals ([2],[3]). 4. An iterative algorithm for discrete extrapolation of band-limited infinite-extent discrete signals (and the minimum norm property of the extrapolation obtained by the iteration [4]) and 5. A certain iterative procedure for extrapolation of band-limited periodic discrete signals [5]. The Bialy algorithm also generalizes the Papoulis-Gerchberg iteration to cases where the ideal low-pass operator is replaced by some other operators.

In the second part of the paper, a suitable modification of this general iteration is shown. This technique leads us to new iterative algorithms for band-limited signal extrapolation. In numerical simulations some of these algorithms provide a fast reconstruction of the sought signal.

I. Introduction

Iterative reconstruction of distorted signals has received much attention in the engineering literature. Many algorithms have been presented for different models of signals. The reader is refered to [6] for a comprehensive review.

In this paper, we will present an approach which unifies a number of important algorithms in the restoration of linearly distorted signals. The basic tool which we will use is that of iterative solution of linear operators in Hilbert spaces. The advantages of this approach, which is based on a result given by Bialy [1], are the following:

1. Several apparently disconnected algorithms, some of which have recently received much interest, can be considered special cases of Bialy's iteration.

2. All these algorithms can be shown to be convergent using a rather general tool.

3. A simple generalization of the basic iterative procedure will be shown to provide some new restoration algorithms which perform fast reconstruction of the sought signal.

Section II reviews some fundamentals of linear operators in Hilbert spaces. Special emphasis is put on pseudoinverse solutions and Bialy's iteration for non-negative symmetric operators. In Section III, we show that this iteration can be used to obtain the iterative procedures mentioned in the abstract of this paper. In particular, we obtain a generalization of the Papoulis-Gerchberg algorithm for the continuous extrapolation problem. In Section IV, we show how a simple generalization of Bialy's iteration provides

some useful recursive techniques for restoration. Some numerical examples showing the performance of these algorithms are presented in Section V, where the application problem is continuous band-limited signal extrapolation. A numerical comparison of these algorithms with the Papoulis-Gerchberg procedure is presented.

II. Basic theory

Let us recall what is meant by bounded and compact linear operators in Hilbert spaces. Let H_1, H_2 be two Hilbert spaces and $A: H_1 \rightarrow H_2$ a linear operator. We say that A is bounded (also continuous) if there exists a real number C such that

$$||A \, x||_2 \leq C \, ||x||_1 \quad \text{for all } x \in H_1$$

where $|| \quad ||_i$ denotes the norm in H_i.

The operator A is called compact if it maps every bounded set $S \subseteq H_1$ onto a set $A(S)$ whose closure $\overline{A(S)}$ is compact. In other words, A is compact if and only if for every bounded sequence $\{x_n, n \in N\} \subseteq H_1$, there exists a subsequence $\{x_{n_k}, k \in N\}$ and $y \in H_2$ such that $A(x_{n_k}) \xrightarrow{||\cdot||_2} y$, $k \rightarrow \infty$. The reader is refered to [7] for further theoretical details.

Obviously, if a linear operator is compact it will also be bounded. The converse does not hold in general. However, if H_2 is of finite dimension both class of operators coincide. The adjoint of A is another linear operator $A^t: H_2 \rightarrow H_1$ characterized by the following identity:

$$\langle A^t y, \; x \rangle_{H_1} = \langle y, \; Ax \rangle_{H_2}$$

where $\langle \; , \; \rangle_{H_i}$ denotes the inner product. A linear operator $A: H \rightarrow H$ is called symmetric if $A = A^t$. In that case, we say that A is non-negative if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. We concern ourselves with iterative solutions to the linear problem $Ax = y$, where $A: H_1 \rightarrow H_2$ is bounded and $y \in H_2$ is given.

Frequently, it happens that y does not belong to the range of A and therefore, thre is no x: Ax = y. In that case, one may attempt to find the minimum norm least squares solution. However, for infinite-dimension spaces this approach is not always successful because the "least square" solutions may fail to exist. We need to recall some related results for our applications.

It is well-known that the range of a bounded operator may not be closed. The situation is even worse for compact operators since it can be proved that the range of such an operator is "almost never" closed. Undoubtedly, this result is the main drawback for a pseudoinverse approach to solving the operator equation Ax = y, because most of distortion equations in signal processing are given by compact operators. The following lemmas, which are proved in ref. [8], help in understanding the matter, and will be useful for the remainder of our paper.

Lemma 1

Let $A: H_1 \to H_2$ be a linear bounded operator. For a fixed $y \in H_2$, let $S = \{x \in H_1: Ax = Qy\}$ and $N = \{x \in H_1: A^t Ax = A^t y\}$. Then $S = N$. ($Q: H_2 \to \overline{A(H_1)}$ is the projection operator onto the closure of the range of A.) The equation $A^t Ax = A^t y$ is recognized as the normal equation for A. It is obvious that if Qy does not belong to the range of A: R(A) then $N = S = \phi$. Therefore, since R(A) may not be closed, there exists many points $y \in H_2: Qy \notin R(A)$. In other words, N will not be empty iff $y \in R(A) + R(A)^\perp$ (⊥ denotes orthogonal set).

Lemma 2

For a fixed $y \in H_2$, the set of least squares solutions

$$\{u \in H_1: ||Au - y|| = \inf \{||Ax - y||, x \in H_1\}\}$$

coincides with the set of solutions of the normal equation $A^t Au = A^t y$.

From the comments given above, it is clear that the set of least squares solutions will not be empty if and only if $y \in R(A) + R(A)^{\perp}$. In that case, this set will be closed and convex and therefore, there will be an element $u = A^+ y$ which has minimum norm among all which satisfy $A^t Au = A^t y$.

Another simple but very important property is the following:

Lemma 3

If $y \in R(A)$, then $A^+ y$ is the minimum norm solution of the linear equation $Ax = y$.

Lemma 3 says that if a solution to the problem $Ax = y$ exists, then the minimum norm solution will make sense and will coincide with the generalized inverse $A^+ y$. This is a very simple consequence of the fact that $y \in R(A)$ ensures that the normal equation $A^t Ax = A^t y$ has the same set of solutions of $Ax = y$.

One would like to have pseudoinverse solutions for every $y \in H_2$. However, as we have shown above this will be possible iff $R(A)$ is closed. In that case, the generalized inverse $A^+: H_2 \rightarrow H_1$ is a well-defined bounded operator. The boundeness of A^+ shows that finding a pseudosolution $A^+ y$ is a stable problem, i.e. small perturbations in the data will produce small changes in the pseudosolution $A^+ y$. As we have mentioned above, this will be "almost never" the case if A is compact:

Lemma 4

If $A: H_1 \rightarrow H_2$ is compact and $R(A)$ is closed then A is degenerate, i.e. $R(A)$ is of finite dimension.

Some examples of compact operators may clarify the matter. Let us suppose that our distortion can be written either as

$$g(y) = \int_{-a}^{a} h(x,y) f(x) dx, \quad y \in (-b,b) \quad (\text{for all } f: \int_{-a}^{a} |f(x)|^2 dx < \infty)$$

for a continuous model or as

$$g(m) = \sum_{n \in Z} h(m,n) f(n), \quad m \in Z \quad (\text{for all } f: \sum_{n \in Z} |f(n)|^2 < \infty)$$

for a discrete model.

In both cases, under rather general conditions on h it can be shown that the corresponding distortion operator is compact. Sometimes, the situation is even worse because the range is not only non-closed but also dense (i.e. $\overline{R(A)} = H_2$). In practical terms, this means that if the given data g is contaminated with some additive noise η, the problem becomes intractable from a generalized inverse point of view. This is because $R(A)^{\perp} = \{0\}$ and therefore, $A^+ g$ will never exist if the noise has any component which is outside of $R(A)$ (this is almost always the case). An example of dense range is provided by the set of band-limited functions, with a fixed bandwidth Ω. It is well known that this set is dense in the set of finite energy functions over an interval $[-a,a]$ (see [9],[2]). We hope to shed more light on this problem in section III.2.

In what follows we will state the Bialy iteration which is also useful for computing generalized inverse solutions of $Ax = y$. This iteration is the main core of the next section, and provides the basic tool for the announced

unification of algorithms.

To this end, if A is a bounded linear operator, we denote by $||A||$ the infimum of the numbers c: $||Ax|| \leq c||x||$, for all $x \in H_1$. We also denote by P the orthogonal projection onto the kernel of A: $Ker(A) = \{x \in H_1: Ax = 0\}$.

Theorem 1 (Bialy, [1])

Let A: H -> H be a linear bounded non-negative operator. For $y \in H$, $x_0 \in H$ consider the iterative process

$$x_{n+1} = x_n + \alpha(y - Ax_{n-1}) \tag{1}$$

where $0 < \alpha < 2/||A||$.

Then, the sequence $\{x_n, n \geq 0\}$ converges if and only if $Ax = y$ has a solution. In that case $x_n \underset{n->\infty}{->} Px_0 + \dot{x}$, where $\dot{x}$ is the minimum norm solution.

We would like to make some remarks about Theorem 1. It is clear that if the initial approximation x_0 is zero then $\{x_n\}$ will approach the minimum norm solution of the equation $Ax = y$. The theorem also says that this will happen iff the equation has at least one solution.

Judging from the appearance of (1) it may be said that recursion (1) tries to compute a fixed point of the mapping $Gx = \alpha y + (I - \alpha A)x$. However, it cannot be said that the fixed point and/or the iterative procedure make sense because of a contractive property of G. In fact, this situation will almost never occur. The reason of this assertion is given by the following.

Lemma 5

If A: H -> H is a bounded linear operator such that iteration (1) converges for all $y \in H$ for some x_0, and $\alpha \neq 0$ then A cannot be compact, unless H

is of finite dimension.

The proof of this lemma is relegated to the reader. Lemma 5 says that if A is a compact operator and the dimension of H is not finite (and therefore, including most of the cases we are interested in) recursion (1) must be divergent for some y. In particular, $I - \lambda A$ will <u>not</u> be a contraction mapping irrespective of the choice of λ.

A relevant characteristic of the hypotheses of Theorem 1 is that A is assumed to be non-negative, excluding apparently many operators for which this condition is not met. However, Bialy's theorem can be used to compute iteratively the minimum norm least squares solution of any bounded linear operator. This result will be obtained very easily if we recall that the minimum norm least squares solution (whenever exists) is the minimum norm solution of the normal equation $A^t A x = A^t y$. Then, Bialy's theorem can be applied because $A^t A$ is a non-negative linear bounded operator. Thus, we have:

Theorem 2

Let $A: H_1 \rightarrow H_2$ be a bounded linear operator. $y \in R(A) + R(A)^\perp$, consider the iterative equation:

$$x_o = 0$$
$$x_n = x_{n-1} + \alpha A^t (y - A x_{n-1}), \quad n \geq 1 \qquad (2)$$

where $0 < \alpha < 2/||A^t A||$. Then $\{x_n\}$ converges to the minimum norm least squares solution $A^+ y$.

It is worth noticing that Theorem 2 assumes $y \in R(A) + R(A)^\perp$ and therefore, the sought generalized inverse solution $A^+ y$ exists.

To end this section, we would like to point out some results which are connected to those of Theorem 1 and Theorem 2. Two special cases of Bialy's theorem were proved earlier for integral operators, which are a case of compact operators arising very often in practical applications. In 1956, Fridman ([10]) proved Theorem 1 for the case: A is given by $Af(x) = \int_{-a}^{a} h(x,t)f(t)dt$, f is any finite energy function, i.e. $f \in L^2(-a,a)$, and the kernel h is positive (and symmetric: $h(x,t) = \overline{h(t,x)}$). In 1951, Landweber ([11]) proved Theorem 2 for the case: $Af(x) = \int_{-a}^{a} h(x,t)f(t)$, removing the assumptions made on $h(x,t)$. In both cases, $h(x,t)$ must define a compact operator, a condition that is often met.

A final remark is in order; it can be easily obtained from Theorem 2 and the discussion on pseudosolutions presented in this section, that iteration (2) will appraoch the minimum norm solution of the equation $Ax = y$, whenever $y \in R(A)$. To see that we just recall that $\{x \in H_1: A^tAx = A^ty\} = \{x \in H_1: Ax = y\}$ for $y \in R(A)$. In particular, if the solution exists and is unique it will be also obtained by the procedure (2).

III. Applications

In this section we will show several applications of the results discussed in Section II.

III.1 Van Cittert-type algorithms.

We now consider a continuous-continuous deconvolution problem. Let $L^2(B)$ denote the Hilbert space of finite energy functions defined on B, i.e. $L^2(B)$ = $\{f: B \rightarrow R: \int_B |f(t)|^2 dt < \infty\}$. Let h be a function such that the following linear operator is bounded:

$$A: L^2(S) \rightarrow L^2(T)$$

$$f \rightarrow Af(t) = \int_S h(t-s)f(s)ds, \quad t \in T$$

(If T or S is bounded, and h satisfies $\int_S \int_T |h(s-t)|^2 dsdt < \infty$ then A will be bounded. In that case, it can be proved that

$$||Af||_{L^2(T)} \leq \{\int_S \int_T |h(s-t)|^2 dsdt\}^{1/2} ||f||_{L^2(S)}$$

where $||\gamma||_{L^2(B)}$ stands for the norm $\{\int_B |\gamma(t)|^2 dt\}^{1/2}$. Another case for which A is bounded will be obtained if the function h has compact support, that is to say, $h(s) = 0$ if $s \notin C$ where C is a compact set in R^n.)

If $S = T$, J_S denotes truncation to S and h satisfies the additional properties $\int_{R^n} |h(t)|^2 dt < \infty$ and $\hat{h}(\omega) \geq 0$ for all $\omega \in R^n$, where $\hat{h}$ denotes Fourier transform, then A is a non-negative operator. To see this,

$$\langle Af,f \rangle_{L^2(S)} = \int_S (Af)(s).f(s) = \int_{R^n} (h*J_S f)(s)J_S f(s)ds$$

where the symbol * stands for convolution, defined over R^n. By means of Parseval's equality, we obtain $\langle Af,f \rangle_{L^2(S)} = \int_{R^n} (H*J_S f)^{\wedge}(w)\overline{[(J_S f)^{\wedge}](w)}dw$. But

$(h*J_S f)^{\wedge}(w) = \hat{h}(w).(J_S f)^{\wedge}(w)$, then $\langle Af, f \rangle_{L^2(S)} = \int_{R^n} \hat{h}(w).|(J_S f)^{\wedge}(w)|^2 dw$, which is always non-negative.

We now assume that $g(s)$, $s \in S$ is the output of our continuous system defined by A. If we are interested in recovering the input $f(s)$, $s \in S$, we can apply Theorem 1 to obtain a sequence $\{f_n\}$ given by

$$f_o = 0$$

$$f_{n+1} = f_n + \alpha(g - h*J_S f_n) \qquad (3)$$

which converges to the minimum energy signal that produces the output g. Another way of writing equation (3) is

$$f_{n+1}(s) = f_n(s) + \alpha(g(s) - \int_S h(s-t)f_n(t)dt), \quad s \in S,$$

or equivalently, $f_{n+1}(s) = \alpha g(s) + (f_n(s) - \alpha \int_S h(s-t)f_n(t)dt)$.

This is a Van Cittert-type recursion whose convergence is ensured. Several remarks are in order. Perhaps the most important observation is that h may have zero frequencies without affecting the convergence of the procedure. It is also clear that many choices of α can be tried whenever $0 < \alpha < 2/||A||$ (if S is bounded we can chose any α which satisfies

$$0 < \alpha < 2/\{\int_S \int_S |h(x-t)|^2 dxdt\}^{1/2}).$$

The classical Van Cittert's algorithm is for the case $S = T = R^n$. It is this assumption what makes the proof of Van Cittert's iteration (3) so simple if α is chosen to satisfy $|1 - \alpha\hat{h}(w)| < 1$ whenever $\hat{h}(w) \neq 0$ ([6]). Therefore, if $S \neq R^n$, under the more stringent condition $\hat{h}(w) \geq 0$, Bialy's iteration provides a non-trivial extension of the classical version of Van Cittert's algorithm.

We next consider a discrete-discrete deconvolution problem. In this case, the underlying Hilbert space is $l_2(B) = \{a_m: m \in B, \sum_{m \in B} |a_m|^2 < \infty\}$, where B is a subset of Z^n. Let $h(m)$ be a sequence such that the operator

$$A: l_2(B) \rightarrow l_2(C)$$

$$x(m), m \in Z^n \rightarrow \sum_{m \in B} h(k-m)x(m), k \in C$$

is bounded. Several conditions on h, similar to those given for the continuous case, can be found to ensure the boundness of A.

If h satisfies $\sum_{m \in Z^n} |h(m)|^2 < \infty$, B = C and the Fourier series of h: $\sum_{m \in Z^n} h(m) e^{-2\pi m \omega} \geq 0$ for all ω, then Bialy's theorem will apply. Thus, the iteration

$$x_m^{(k)} = x_{m-1}^{(k)} + \alpha(g(k) - h*J_B x_{m-1}(k)), \quad k \in B \tag{4}$$

will converge to the minimum norm solution of the problem: $g(m) = (h*J_B f)(m)$, $m \in B$, provided that at least one solution exists. Equation (4) can be written as follows

$$x_m(k) = \alpha g(k) + x_{m-1}(k) - \alpha \sum_{j \in B} h(k-j) x_{m-1}(j), \quad k \in B \tag{4'}$$

with $0 < \alpha < 2/||A||$. A simple rationale for choosing α is $0 < \alpha < 2/(\sup|\hat{h}|)$.

Equation (4) (or its equivalent form (4')) is a Van-Cittert's recursive formula, when the model for the observed and unknown signals are both discrete.

It is worth noticing that equation (4') can also be written by using an operator-type notation:

$$x_m^{(k)} = \alpha g(k) + \{(\delta - \alpha h)^* J_B x_{m-1}\}(k), \ k \in B, \tag{5}$$

where $\delta: Z^n \to C: \delta(0) = 1, \ \delta(k) = 0$ if $k \neq 0$.

Recursion (5) was also considered in [6] for the special case in which B is bounded. Under this assumption, equation (5) was shown to converge in ref. [12]. However, in [12] it was also proved that if B is bounded α can be chosen independently of B so that $(\delta - \alpha h)^* J_B$ is a contraction mapping under mild assumptions on h (which include the case $\hat{h}(\omega) \geq 0$ for all $\omega \in R^n$). We think that it is now understood what is the theoretical importance of the a priori constraint that the sought signal is limited to the set B. From lemma 5 and related discussions, it is seen that $(\delta - \lambda h)^* J_B$ will be a contraction mapping for a certain λ and for a rather general h only if B is bounded. On the other hand, if B is not bounded, equation (5) was shown to converge to the minimum norm solution of the deconvolution problem (Theorem 1), but the contractive property of $(\delta - \lambda h)^* J_B$ will not hold in general.

To conclude this subsection we would like to emphasize that for the continuous-continuous model, if the set S where the input signal f is not zero is bounded, $(I - \alpha h)^* J_S$ will not be a contraction mapping in general. This is a major difference between the continuous and discrete model, $l_2(B)$ is of finite dimension if B is bounded whereas $L^2(S)$ does not have this property.

III.2 Pseudoinverse regularization

The deconvolution problem that was discussed in Section III.1 usually requires a more involved solution due to the following facts:

1. g is given with noise and therefore the solution to the problem $g = Af$ may not exist.

2.   h may not satisfy $\hat{h}(w) \geq 0$ for all w.

3.   The period of time where the observation g is given: C may not coincide with the support B of the sought signal.

A full answer to problems 2 and 3 and a partial solution to 1 will be given in this section. To this end, we will show the convergence of an iterative recon- struction algorithm. We will consider the discrete-discrete model only. Simi- lar comments and results hold for the continuous-continuous case. With the same notation as in Section III.1, let us suppose that our observation g(m), m $\in$ C is given with noise. Assume that $g \in R(A) + R(A)^{\perp}$. Then, the following problem will always have a solution: $A^{t}Af = A^{t}g$ where A^{t} denotes the adjoint of A (see Section II). For the convolution case:

$$(Af)(m) = \sum_{k \in B} h(m-k)f(k),$$

m $\in$ C then, $(A^{t}q)(k) = \sum_{m \in C} \overline{h(-k+m)}q(m)$, k $\in$ B. This means that A^{t} is also given in terms of a convolution where the new kernel is $\overline{h(-m)}$. Specifically, $A^{t}A$ is given by

$$(A^{t}Af)(j) = \sum_{k \in B} (\sum_{m \in C} \overline{h(m-j)}h(m-k))f(k), \quad j \in B$$

which is always non-negative, as it was pointed out in Section 2.

Thus, we can apply Theorem 2 for computing the minimum norm least squares solution of Af = g, if $g \in R(A) + R(A)^{\perp}$, by means of the iteration:

$$x_{0} = 0$$

$$x_{m} = x_{m-1} + \alpha A^{t}(g - Ax_{m-1}) \quad m \geq 1 \tag{6}$$

where $0 < \alpha < 2/(||A||^{2})$.

An equivalent expression for (6) is obtained by replacing A and A^t

$$x_0 = 0$$

$$k \in B, \quad x_m(k) = \alpha \sum_{m \in C} \overline{h(m-k)} g(m) + x_{m-1}(k) - \alpha \sum_{j \in B} (\sum_{m \in C} \overline{h(m-k)} h(m-j)) f(j)$$

$$(6')$$

It is worth pointing out that equation (6) or (6') will not converge if the noisy data $g \not\in R(A) + R(A)^1$. However, we think that this approach is useful for understanding the limitations of the technique and for setting a condition to ensure convergence or divergence of the iteration.

A particular case is obtained for $B = C = Z^n$. In that case, the technique that consists of convolving the equation $h * f = g$ with $\overline{h(-m)}$ has been proposed independently by several authors ([6],[13]) but the approaches used were conceptually different. For $B = C = Z^n$, the operator $A^t A$ is given by a convolution whose kernel is $h(m) * \overline{h(-m)}$. Then, the transfer function of the system $A^t A$ is $|\hat{h}(w)|^2$, $w \in R^n$. Since $|\hat{h}(w)|^2$ is always non-negative, Van-Cittert's algorithm applies to the equation $[\overline{h(-m)} * h(m)] * f = \overline{h(-m)} * g$, obtaining the following frequency-space recursion:

$$\hat{x}_0(w) = 0$$

$$\hat{x}_m(w) = \alpha \overline{\hat{h}(w)} \hat{g}(w) + (1 - \alpha |\hat{h}(w)|^2) \hat{x}_{m-1}(w) \tag{7}$$

which is, of course, equivalent to (6') when $B = C = Z^n$. It is worth pointing that there should be a solution to the problem $h(m) * f = g$ in order to a finite-energy discrete signal whose Fourier transform is the limit of

above, the pseudoinverse approach provides a full answer to and C are any subset of Z^n, and the convergence of (6) is

characterized in terms of the data g.

To end this subsection we would like to remark that if $g \notin R(A) + R(A)^{\perp}$ then (6) will not converge. Therefore, caution is recommended when the iteration (6) (or (6')) is used for theoretical derivations. On the other hand, when (6) (or (6')) is implemented numerically, a finite piece of the signals is used and therefore convergence of the iteration is guaranteed. The conceptual point is that for implementation purposes, the underlying model for the distortion is $g(m) = (h*J_B f)(m)$, $m \in C$ where both B and C are finite. Then, the pseudoinverse solution will always exist and can be approximated by means of (6').

III.3 Papoulis-Gerchberg's iteration

Let us assume that $g: F \rightarrow C$ is a piece of a Ω-band-limited function (i.e. $\hat{g}(w) = 0$, $w \notin \Omega$) where $F \neq \emptyset$ is an open subset of R^n. Let us suppose that the complete function g satisfies the finite-energy constraint: $\int_{R^n} |g(x)|^2 dx < \infty$.

Papoulis [2] and Gerchberg [3] proposed the following algorithm for computing the continuation of g or its Fourier transform:

$$g_0 = 0$$

$$g_m = \text{sinc}_{\Omega} * (J_F g + (I - J_F) g_{m-1}), \quad m \geq 1 \tag{8}$$

where sinc_{Ω} denotes the function whose Fourier transform is the indicator of Ω.

In ref. [2], equation (8) was shown to be convergent to g in the energy norm for the one-dimensional case. In ref. [14], another approach was shown to prove convergence of (8) which is also valid for the multi-dimensional case.

However, in [15], this algorithm was presented as a special case of Landweber's iteration ([11]). The underlying operator equation is $(Af)(x) = g(x)$ where A is an integral operator given by

$$(Af)(x) = \int_{\Omega} f(w) e^{-2\pi i w x} dw, \quad x \in F \tag{9}$$

$f \in L^2(\Omega)$ and Ω is assumed to be bounded. It is obvious that the sought solution is $f = \hat{g}$ and is also unique. We can now apply Theorem 2 to get a recursion:

$$f_o = 0$$
$$f_m = f_{m-1} + \alpha A^t(g - Af_{m-1}) \tag{10}$$

where $0 < \alpha < 2/(||A^tA||)$ and A^t is the adjoint integral operator given by $(A^th)(w) = \int_F h(x) e^{2\pi i w x} dx, \quad w \in \Omega.$

It is very easy to verify that $||A^tA|| \leq 1$; then, $\alpha = 1$ is an admissible value and from (10)

$$f_o = 0$$
$$f_m = f_{m-1} + A^t(g - Af_{m-1}) \tag{10'}$$

will converge in the energy norm to the unique solution f of the equation $(Af)(x) = g(x), \quad x \in F$. But in that case, $\overset{\vee}{f}_m \underset{m \to \infty}{\to} \overset{\vee}{f}$ also in the energy norm ($\vee$ denotes inverse Fourier transform). Since f_n, f are supported on Ω, $\overset{\vee}{f}_n(x) = \int_{\Omega} f_n(w) e^{-2\pi i x w} dw$ and $\overset{\vee}{f}(x) = \int_{\Omega} f(w) e^{-2\pi i w x} dw, \quad x \in R^n$. We now apply inverse transform to both sides of (10'):

$$\overset{\vee}{f}_o = 0$$
$$\overset{\vee}{f}_m = \overset{\vee}{f}_{m-1} + \{\int_F (g(x) - \int_{\Omega} f_{m-1}(w) e^{-2\pi i x w} dw) e^{2\pi i x \cdot} dx\}^{\vee} \tag{11}$$

If we call $g_m = \overset{\vee}{f}_m$, $m \geq 0$ we will obtain the recursion

$$g_0 = 0$$

$$g_m = g_{m-1} + \text{sinc}_\Omega * (J_F g - J_F g_{m-1}) \tag{11'}$$

Since g_{m-1} is Ω-band-limited, equation (11') is equivalent to the following:

$$g_0 = 0$$

$$g_m = \text{sinc}_\Omega * (J_F g + (I-J_F) g_{m-1}), \quad m \geq 1 \tag{12}$$

Equation (12) is the Papoulis-Gerchberg iteration (8).

We will now show a generalization of (8) to cases where the low-pass convolution is performed by some other operator.

To this end, we need to assume some further information related to $\hat{g} = f$. Let us suppose that for certain non-negative bounded function $\hat{h}(w)$, $w \in \Omega$, $\hat{h}(w) = 0$, $w \notin \Omega$, g satisfies:

$$\int_\Omega \frac{|g(w)|^2}{\hat{h}(w)} \, dw < \infty \tag{13}$$

Then, if we consider the operator $A: L^2(\Omega) \rightarrow L^2(F)$:

$$(As)(x) = \int_\Omega \hat{h}^{1/2}(w) . e^{2\pi i w x} s(w) \, dw, \quad x \in F \tag{14}$$

the equation $g(x) = (As)(x)$, $x \in F$ will have a solution in $L^2(\Omega)$ (which is obviously $g/\hat{h}^{1/2}$). It readily follows that the solution is also unique. We can now apply Theorem 2 to the equation $g(x) = (As)(x)$, $x \in F$ for A given in (14) to get an iterative procedure:

$$f_0 = 0$$

$$f_m = f_{m-1} + \alpha A^t (g - A f_{m-1}), \quad m \geq 1 \tag{15}$$

which converges to the unique solution. More specifically equation (15) becomes

$$f_o = 0$$

$$f_m(w) = f_{m-1}(w) + \alpha \int_F \hat{h}^{1/2}(w) e^{2\pi i w x}(g(x) - \int_\Omega \hat{h}^{1/2}(z) e^{-2\pi i x z} f_{m-1}(z) dz) dx$$

$$(15')$$

If we now multipy both sides of (15') by $\hat{h}^{1/2}(w) e^{-2\pi i w y}$, integrate respect to $w \in \Omega$, and call $g_m(y) = \int_\Omega f_m(w) \hat{h}^{1/2}(w) e^{-2\pi i w y} dw$, we will obtain

$$g_o = 0$$

$$y \in R^n, \quad g_m(y) = g_{m-1}(y) + \alpha \int_F \{\int_\Omega \hat{h}(w) e^{2\pi i w(x-y)} dw\}(g(x) - g_{m-1}(x)) dx \quad (16)$$

If we call $h(z) = \int_\Omega \hat{h}(w) e^{2\pi i w z} dw$, equation (16) will become

$$g_o = 0$$

$$g_m = g_{m-1} + \alpha h(-z) * J_F(g - g_{m-1}), \quad m \geq 1 \quad (17)$$

which converges to g uniformly over compact sets in R^n, when $0 < \alpha < 2/(\sup_w |\hat{h}(w)|)$ and (13) is satisfied.

Two well-known discrete algorithms for extrapolation can be considered a sampled version of equations (8) and (11') (see [16],[17]). In addition, some new algorithms for solving the discrete extrapolation problem ([18]) can also be interpreted as a sampled approximation of equation (17).

III.4 Iterative extrapolation of infinite-extent discrete signals.

Let F be a finite subset of Z^n and $z(m)$, $m \in F$ a sequence of numbers. The discrete band-limited extrapolation problem consists of finding an infinite sequence $y(m)$, $m \in Z^n$ such that $y(m) = z(m)$, $m \in F$ and $y(m)$ is Ω-band-limited, i.e. $y(w) = \sum_{m \in Z^n} y(m) e^{-2\pi i m w} = 0$ if $w \notin \Omega$ (a fixed bounded set of

frequencies $\Omega \subseteq [-1,1]^n$ (see [4],[16],[17],[18]).

The solution to this problem is non-unique ([6],[4]). In ref. [4] it was shown that the minimum-norm discrete extrapolation y can be computed by means of the following two-step procedure:

1. solve for x: $\displaystyle\sum_{m \in F} \text{sinc}_\Omega(k-m)x(m) = z(k), \; k \in F$ (18a)

2. compute $\displaystyle\sum_{m \in F} \text{sinc}(k-m)x(m) = y(k), \; k \in Z^n$ (18b)

Then, it was shown that y can be computed by the following iterative algorithm:

$$y_0 = 0$$

$$k \in Z^n, \; y_m(k) = y_{m-1}(k) + \alpha \sum_{j \in F} \text{sinc}_\Omega(k-j)(z(j)-y_{m-1}(j)), \; m \geq 1 \quad (19)$$

for $0 < \alpha < 2$. (Both results were extended for arbitrary multidimensional F and Ω in [18]; for a relationship between this discrete solution and the continuous extrapolation problem given in III.3, see [18].)

Perhaps, the earliest reference to the technique given by (18a)-(18b) is Yao [19] who addressed this problem under a rather different name and by using a quite general approach.

The fact that iteration (19) computes the same sequence as that of (18a)-(18b) is very simple. In this section, we will show that the iteration (19) can be obtained from Bialy's iteration for a certain operator equation problem. The minimum norm property of the limit sequence will be readily derived as a byproduct.

Let $A: L^2(\Omega) \rightarrow l_2(F)$ be the following linear operator:

$$(Af)(m) = \int_{\Omega} f(w) e^{2\pi imw} dw, \quad m \in F$$

It is clear that A is bounded when $L^2(\Omega)$ and $l_2(F)$ are equipped with the norms $\int_{\Omega} |f(w)|^2 dw$ and $\sum_{m \in F} |z(m)|^2$ respectively. It is clear that the discrete extrapolation problem can be put in this equivalent way:

$$\text{find } f \in L^2(\Omega): (Af)(m) = z(m), \quad m \in F \tag{20}$$

From Parseval's formula, it is seen that the minimum norm extrapolation corresponds to minimizing $||f||_2$ where f satisfies (20). We can now solve (20) by means of Bialy's iteration. To this end, we need to compute A^t. It is simple to verify that, if $s \in l_2(F)$ then

$$(A^t s)(w) = \sum_{m \in F} s(m) e^{-2\pi imw}, \quad w \in [-1,1]^n.$$

Thus, Bialy's iteration given by Theorem 2 becomes

$$f_o = 0 \tag{21}$$

$$f_m(w) = f_{m-1}(w) + \alpha \sum_{k \in F} (z(k) - \int_{\Omega} f_{m-1}(z) e^{2\pi izk} dz) e^{-2\pi ikw}, \quad w \in \Omega.$$

and f_m converges to the minimum norm solution of (20) in the $L^2(\Omega)$ norm. Therefore, $\int_{\Omega} f_m(w) e^{2\pi iwk} dw, k \in Z^n$ approaches the minimum norm Ω-band-limited extrapolation $y(k)$, $k \in Z^n$ when $m \to \infty$ in the $l_2(Z^n)$ norm (Parseval's formula). Then, if we call $y_m(k) = \int_{\Omega} f_m(w) e^{2\pi ikw} dw, k \in Z^n$, equation (21) becomes

$$y_o = 0$$

$$k \in Z^n, \quad y_m(k) = y_{m-1}(k) + \alpha \sum_{j \in F} sinc_{\Omega}(k-j)(z(j) - y_{m-1}(j)), \quad m \geq 1 \tag{19}$$

and convergence to $y(k)$, $k \in Z^n$ is ensured for $0 < \alpha < 2$.

A final remark is in order. The operator A given by (20) satisfies $R(A) = l_2(F)$ because F is finite, therefore iteration (21) or equivalently (19) will always converge to the minimum-energy solution of the problem. This means that the algorithm does not distinguish signal from noise.

III.5 Iterative extrapolation of periodic discrete signals.

Another related discrete approach to band-limited extrapolation is to solve the following problem:

Given $z(k)$, $k = -k_o,\ldots,k_o < N$

Find $y(k)$, $-N \leq k \leq N$:

$$y(k) = z(k), \quad -k_o \leq k \leq k_o$$

$$\sum_{k=-N}^{N} y(k)e^{-2\pi ikn/M} = 0, \quad |n| > k_o \qquad (22)$$

where $M = 2N + 1$.

In this case, the band-limited property of $y(k)$, $-N \leq k \leq N$ is given in terms of the discrete Fourier transform (DFT): $\sum_{k=-N}^{N} y(k)e^{-2\pi ikn/M}$.

In ref. [5] the following iterative algorithm for computing the extrapolation (22) was shown to be convergent:

$$y_o(k) = 0, \quad -N \leq k \leq N$$

$$y_n = \text{IDFT} \begin{cases} \beta_n(k), & -k_o \leq k \leq k_o \\ 0, & \text{otherwise} \end{cases} \qquad (23a)$$

$$\text{where } \beta_n = \text{DFT} \begin{cases} z(k), & -k_o \leq k \leq k_o \\ y_n(k), & |k| > k_o \end{cases} \qquad (23b)$$

(IDFT stands for the inverse discrete Fourier transform given by
$1/M \cdot \sum_{k=-N}^{N} x(k) e^{2\pi i k n/M}$, $n = -N, \ldots, N$).

It is clear that procedure (23) incorporates at every iteration the information available in both time and frequency domains. In ref. [5], the proof of the convergence of this recursion was done by means of a certain nonexpansive property of an operator in C^M.

In this section, we show that (23) can be also considered a special case of Bialy's theorem. Perhaps, this is the simplest of the examples presented in this section because of the finite dimensional nature of the underlying Hilbert spaces.

Specifically, let $A: C^{2k_0+1} \rightarrow C^{2k_0+1}$ given by the IDFT operator:

$$(Ax)(n) = \frac{1}{M} \sum_{k=-k_0}^{k_0} x(k) e^{2\pi i k n/M}, \quad -k_0 \leq n \leq k_0$$

It is obvious that problem (22) can be restated as that of finding a vector $x \in C^{2k_0+1}$ such that $(Ax)(k) = z(k)$, $-k_0 \leq k \leq k_0$. It is known that this system of equations has a unique solution. We can apply Bialy's iteration (2) for computing the solution x. So, we obtain

$$x_0(k) = 0, \quad -k_0 \leq k \leq k_0$$
$$x_n = x_{n-1} + \alpha A^t(z - Ax_{n-1}), \quad n \geq 1 \tag{24}$$

where α can be chosen as 1. (Here, A^t is transpose-conjugate of A.)

We now take M-length IDFT on both sides of (24) to obtain

$$y_0 = 0$$

$$y_n(k) = y_{n-1}(k) + \frac{1}{M} \sum_{h=-k_o}^{k_o} e^{2\pi ihk/M} \sum_{m=-k_o}^{k_o} e^{-2\pi ihm/M} (z(m)-y_{n-1}(m)) \quad (25)$$

$$-N \leq k \leq N,$$

It is easy to verify that (25) can also be written in the following way:

$$y_o = 0$$

$$y_n(k) = \frac{1}{M} \sum_{k=-k_o}^{k_o} e^{2\pi ihk/M} \sum_{m=-N}^{N} e^{-2\pi ihm/M} (J_{k_o} z + (I-J_{k_o})y_{n-1})(m) \quad (26)$$

where J_{k_o} denotes truncation to $[-k_o, k_o]$.

It turns out that recursion (26) is the same as (23a)-(23b) and therefore, the convergence of y_n to the sought extrapolation is ensured.

In the derivation presented above it was assumed to simplify notation that the length of the DFT is odd: $2N + 1$.

The advantage of this approach to interpreting iteration (23) is that it is possible to characterize the convergence of a similar procedure when the number of samples in the time and frequency domains is not the same. In such a case, it is obvious that the extrapolation problem has no solutions or an infinite number of solutions. In both cases, the corresponding equation (26) will provide the minimum norm least squares extrapolation.

IV. Extensions

In this section, we show that some extensions of the Bialy iteration can be useful for obtaining new algorithms for signal restoration and extrapolation.

In ref. [20], we presented the following iteration

$$f_o = 0$$

$$f_n = f_{n-1} + DA^t(g - Af_{n-1}), \quad n \geq 1, \tag{27}$$

and we related it to the numerical continuation of analytic functions. Let us assume that $A: H_1 \to H_2$ is a bounded linear operator, $g \in R(A) + R(A)^\perp$ and $D: R(A^t) \to R(A^t)$ is a bounded linear symmetric operator which is assumed to be one-to-one (i.e. $\forall x: Dx = 0$ then $x = 0$) and such that DA^tA is non-negative. Under this assumption it can be easily shown that $\lim_{n \to \infty} f_n = f$ where f is the minimum norm least squares solution of $Af = g$, if D is chosen so that $||DA^tA|| < 2$.

In case A is compact, the condition DA^tA is non-negative may be put in terms of the eigenvectors of A^tA. This case was extensively analyzed in [21]. The effects of different D's on the speed of convergence of iteration (27) was also studied ([21]) for the compact case.

We will next show how iteration (27) can be used for obtaining some generalizations of the Landweber-Papoulis-Gerchberg's algorithm discussed in Section III.3.

To this end let us call $A: L^2(\Omega) \to L^2(F)$ the compact operator given in Section III.3

$$(Af)(x) = \int_\Omega f(w)e^{-2\pi iwx}dw, \quad x \in F \tag{9}$$

Since $A^t A$ is another linear non-negative compact operator there exists a family of eigenvalue-eigenfunction (λ_i, ϕ_i) i=1,2,... of $A^t A$ such that:

$$A^t A \phi_n = \lambda_n \phi_n, \quad n=1,2,\ldots$$

(see [7]).

A sufficient condition on D for ensuring convergence of iteration (27) is the following (see [21]):

a. $D\phi_n = p_n \phi_n$, n=1,2,...

b. p_n satisfies $0 < p_n \lambda_n < 2$ for all n.

c. p_n, n=1,2,... is a bounded sequence.

It is interesting to remark that the operator $A^t A$ is given by the integral kernel: sinc_F and therefore ϕ_n, n=1,2,... are the prolate spheroidal wave functions ([9]).

Many operators can be chosen to satisfy conditions a, b and c. In [21], it was shown that it is sufficient to pick $D = G(A^t A)$ where $G(\lambda)$ is a polynomial or rational function such that $0 < \lambda G(\lambda) < 2$ for $0 < \lambda \leq 1$. If D is to be so chosen, (27) will converge in the $L^2(\Omega)$ norm to the solution of the problem $(Af)(x) = g(x)$, $x \in F$, where $g: R^n \rightarrow C$ is assumed to be a Ω-band-limited function. If we now apply inverse Fourier transform to both sides of (27) we will get the following recursion

$$g_o = 0$$

$$g_n(x) = g_{n-1}(x) + \int_\Omega e^{2\pi i w x} D\{\int_F e^{-2\pi i z} \cdot (g(z) - g_{n-1}(z)) dz\}(w) dw \qquad (28)$$

Observe that when $D = I$, (28) becomes the Landweber Papoulis-Gerchberg algorithm. Equation (28) shows a quite general version of this classical

situation.

In the remainder of this section, we will present some numerical simulation results comparing the generalization (27) with the classical iteration (10'). To this end, let us define

$$g: R \rightarrow R: g(x) = (\frac{\sin \pi/2\ x}{\pi/2\ x})^2 \cos \pi x$$

The Fourier transform of this signal is plotted in Fig. 1. If we take the interval $F = (-1,1)$ as the known part of g, a fairly reasonable reconstruction of the Fourier transform can be obtained by means of Discrete Fourier Transform (DFT) of 129 samples. This result is plotted in Fig. 2. It is clear that the two peaks are easily distinguished. On the other hand, if $F = (-1/2, 1/2)$ the situation will be completely different. Figure 3 plots the result obtained for DFT of 129 samples in $(-1/2, 1/2)$. This means that restricting the known part to $(-1/2, 1/2)$ represents an irretrievable loss for the application of the naive inversion technique. In other words, by means of DFT of samples of g on $(-1/2, 1/2)$ the outstanding features of the spectrum of g are lost. Therefore, we think that $g: [-1/2, 1/2] \rightarrow R$ is a reasonable test example for our numerical simulations.

We first apply the Landweber-Papoulis-Gerchberg iteration. Figure 4a shows the very poor result obtained after 20 iterations. Figure 4b plots the reconstructed Fourier transform after 500 iterations. In this case the

We now apply the more general procedure given by (27) for three different D's.

1. $D = (A^t A + \gamma I)^{-1}$

For this operator (27) is closely related to the Twomey-Tikhonov regularization method. In this case, iteration (27) becomes:

$$f_o = 0$$
$$(A^t A + \gamma I) f_n = \gamma f_{n-1} + A^t g, \quad n \geq 1 \tag{29}$$

It is worth pointing out that γ should be chosen positive. In that case, $A^t A + \gamma I$ is always invertible.

Figure 5a shows the result obtained after 10 iterations when $\gamma = 0.00005$, and Figure 5b plots the reconstructed Fourier transform after 20 iterations with the same parameter γ. In both cases the reconstructions are of good quality.

Fixing a value for γ and determining the number of iterations are by no means trivial matters. By comparing Figures 5a and 5b it is seen that the reconstruction is quite sensitive to the number of iterations d es. We think that the sensitivity depends also on the parameter γ.

Figure 6a shows the result after 10 iterations obtained by applying (29) when $\gamma = 0.005$. Figure 6b plots the corresponding result for $\gamma = 0.005$ and 50 iterations. By comparing Figures 5a and 6a it is seen that the reconstruction is very sensitive to the parameter γ when the number of iterations is fixed.

Figure 7 shows the reconstruction obtained for $\gamma = 0.000005$ after 10 iterations. It is clear that for a fixed number of iterations the smaller the parameter γ is, the more distorted (due to the propagation of round-off errors) the reconstruction will be.

In spite of some unanswered questions, the main conclusion that can be drawn from these examples is that the resolution obtained in Figures 5a, 5b

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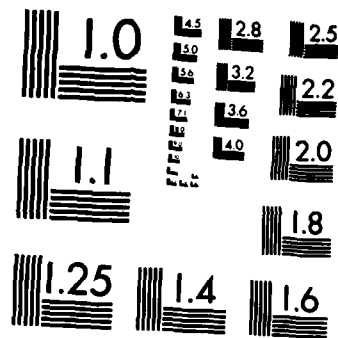
MICROCOPY RESOLUTION TEST CHART
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6a and 7 is much better than that of Figures 4a and 4b.

2. $D = F(A^tA)$ and $F(\lambda) = 804.375\lambda^6 - 3003\lambda^5 + 4504.5\lambda^4 - 3465\lambda^3 + 1443.75\lambda^2 - 315\lambda + 31.5$

Some reasons for choosing such a D are well documented in [21]. We think that this example is also useful to realize that the reconstruction is very sensitive to the choice of D. For this D, Figure 8a shows the result after 10 iterations. It is seen that the resolution is poor. However, Figure 8b plots the reconstructed Fourier transform for 200 iterations which is a good result. This means that the procedure is slower compared to those given where $D = (A^tA + \gamma I)^{-1}$.

It is also remarkable that by using a fewer number of iterations than those necessary for the classical Landweber-Papoulis-Gerchberg algorithm recursion (27) provides a better reconstruction (compare Figures 4b and 8b).

3. $D = F(A^tA)$ and $F(\lambda) = \lambda^{16}$.

This case is intended to be an example where the speed of the reconstruction seems to be similar to that of the classical approach (9)-(10').

Figure 9a shows the reconstruction obtained after 500 iterations. By comparing 9a and 4b is is seen that the results look much the same. Figure 9b plots the result obtained after 1,000 iterations. By comparing 9b with 9a it is noticeable that the reconstruction of the Fourier transform was improved at the cost of double computational effort.

It was assumed, so far, that the given signal is not contaminated with any noise. Since the techniques presented in 1 and 2 above represent a substantial improvement of the classical iteration procedure (8)-(10') it is

expected that the noise will also propagate much faster in the reconstruction. Therefore, a stopping rule is of great importance for practical applications. It is also important to analyze what is the performance of the iteration when the known range of g is smaller. Some related examples and further analyses are given in [23].

Acknowledgements

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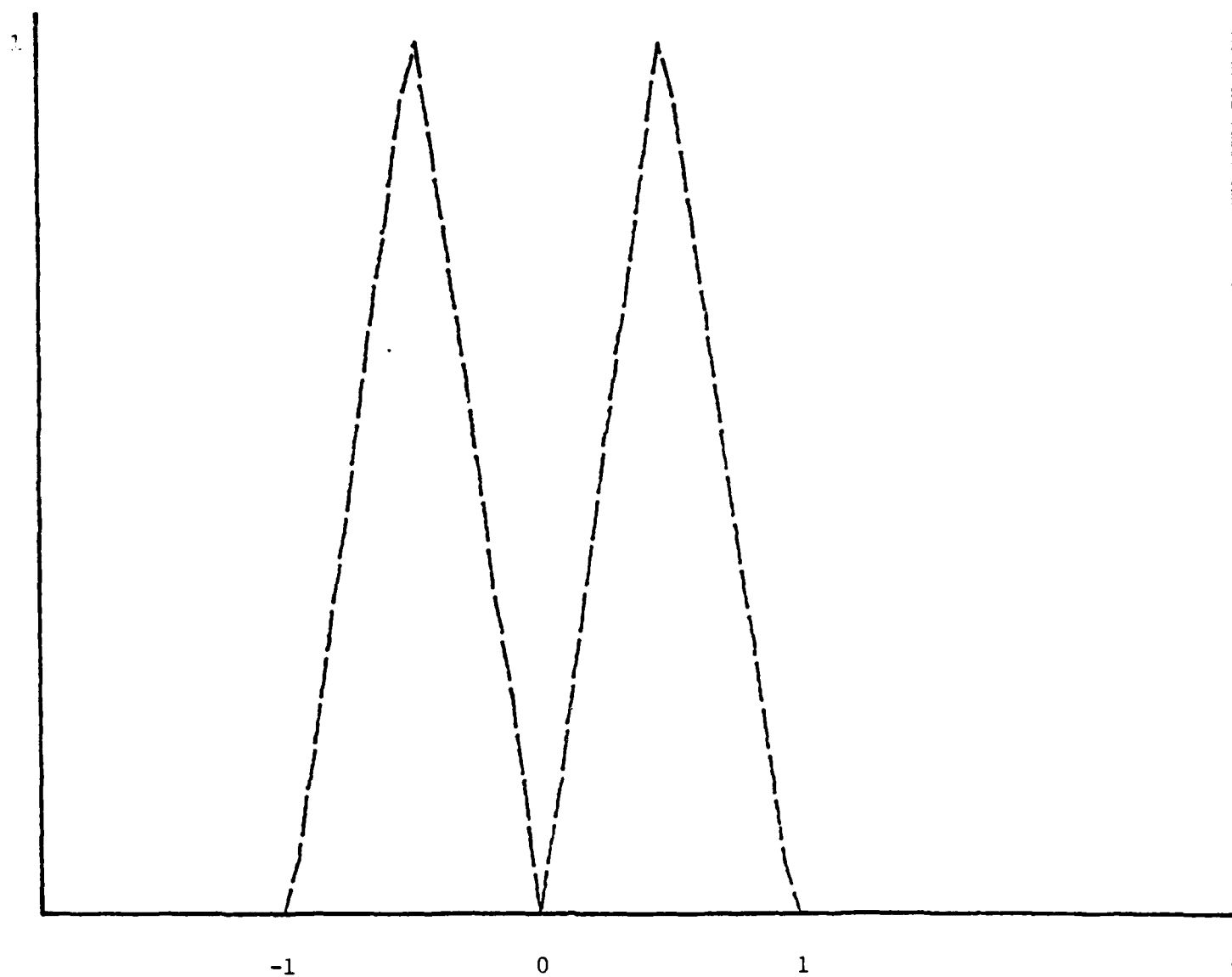
Figure 1.   Fourier transform of the ideal signal.
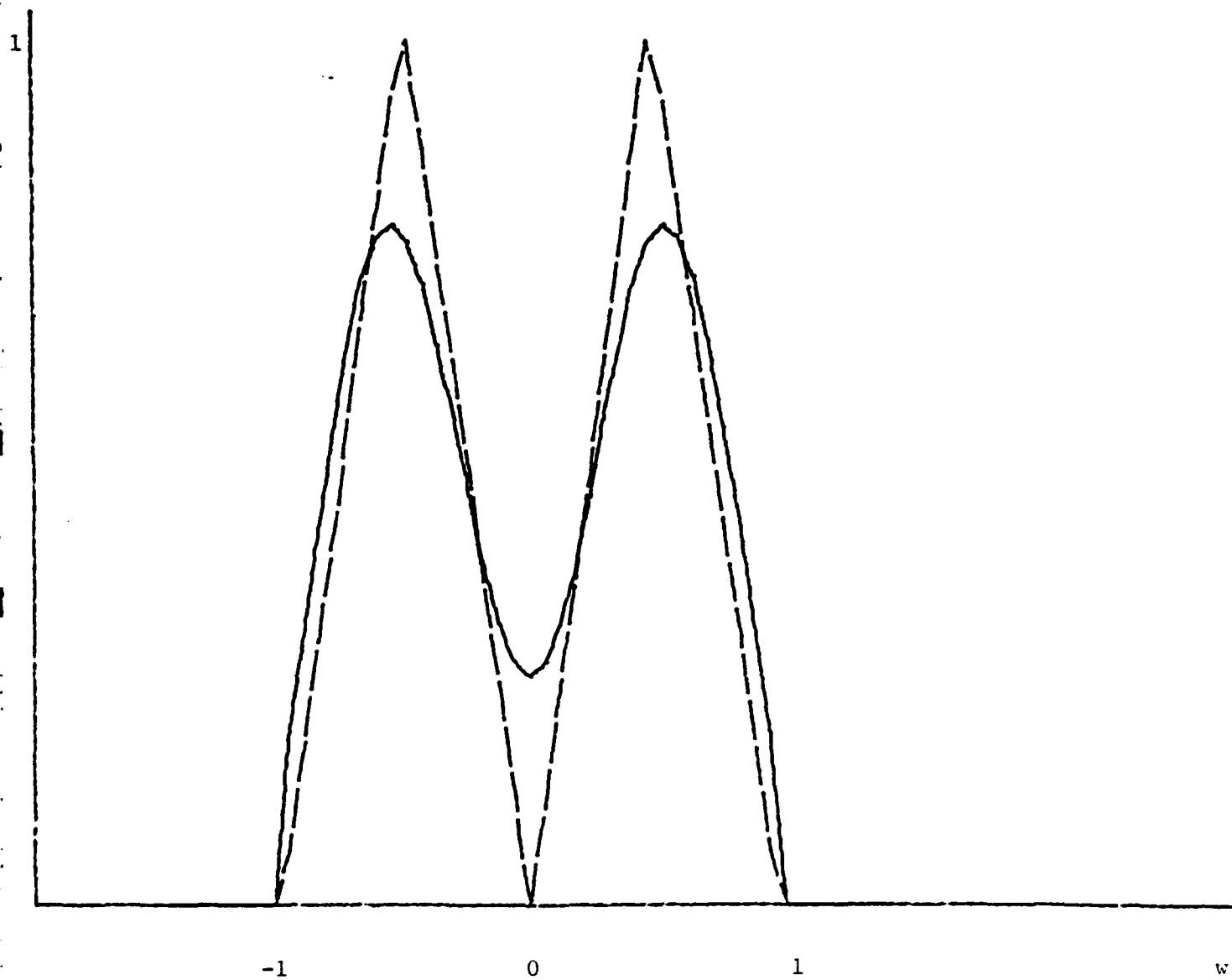
Figure 2.   DFT of 129 samples in (-1,1).
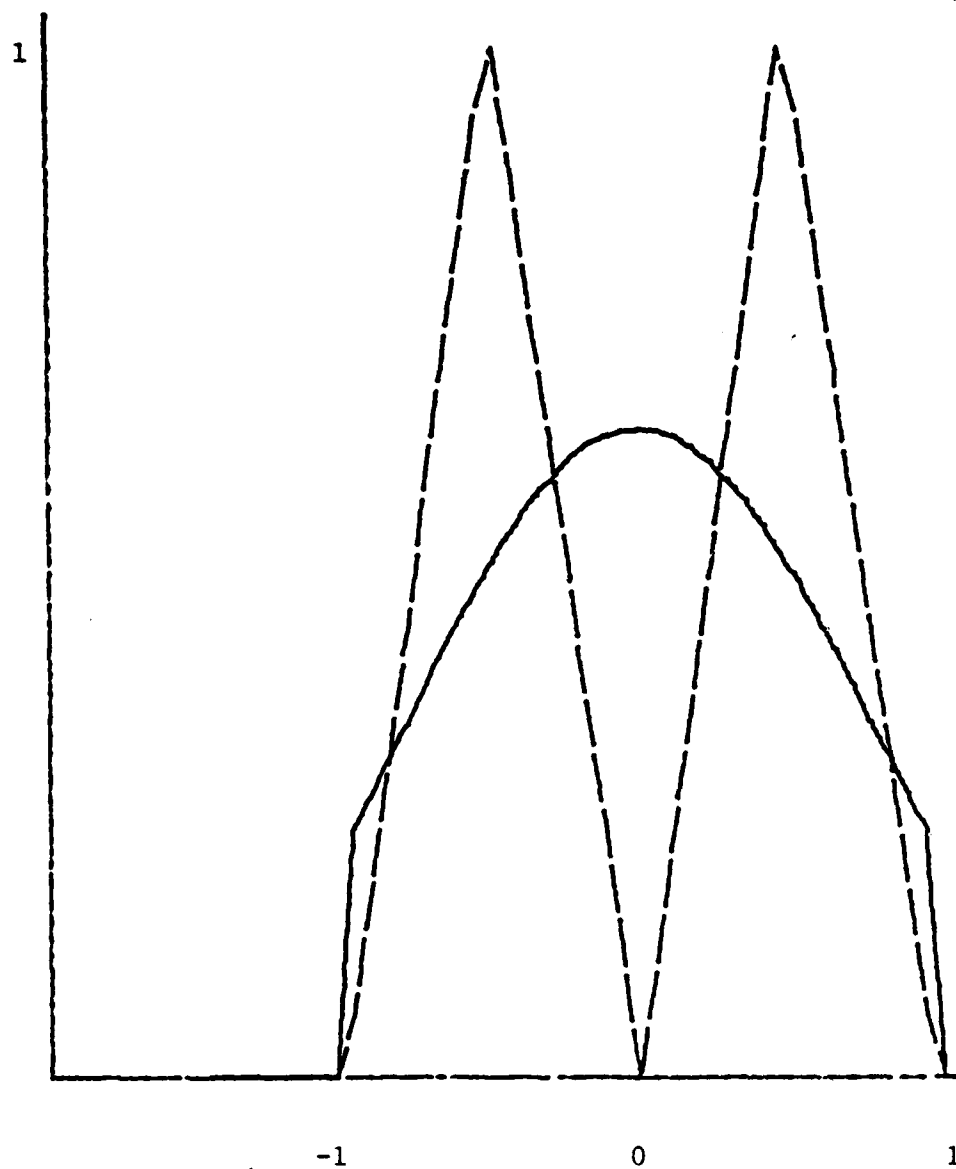
Figure 3.   DFT of 129 samples in (-1/2, 1/2).
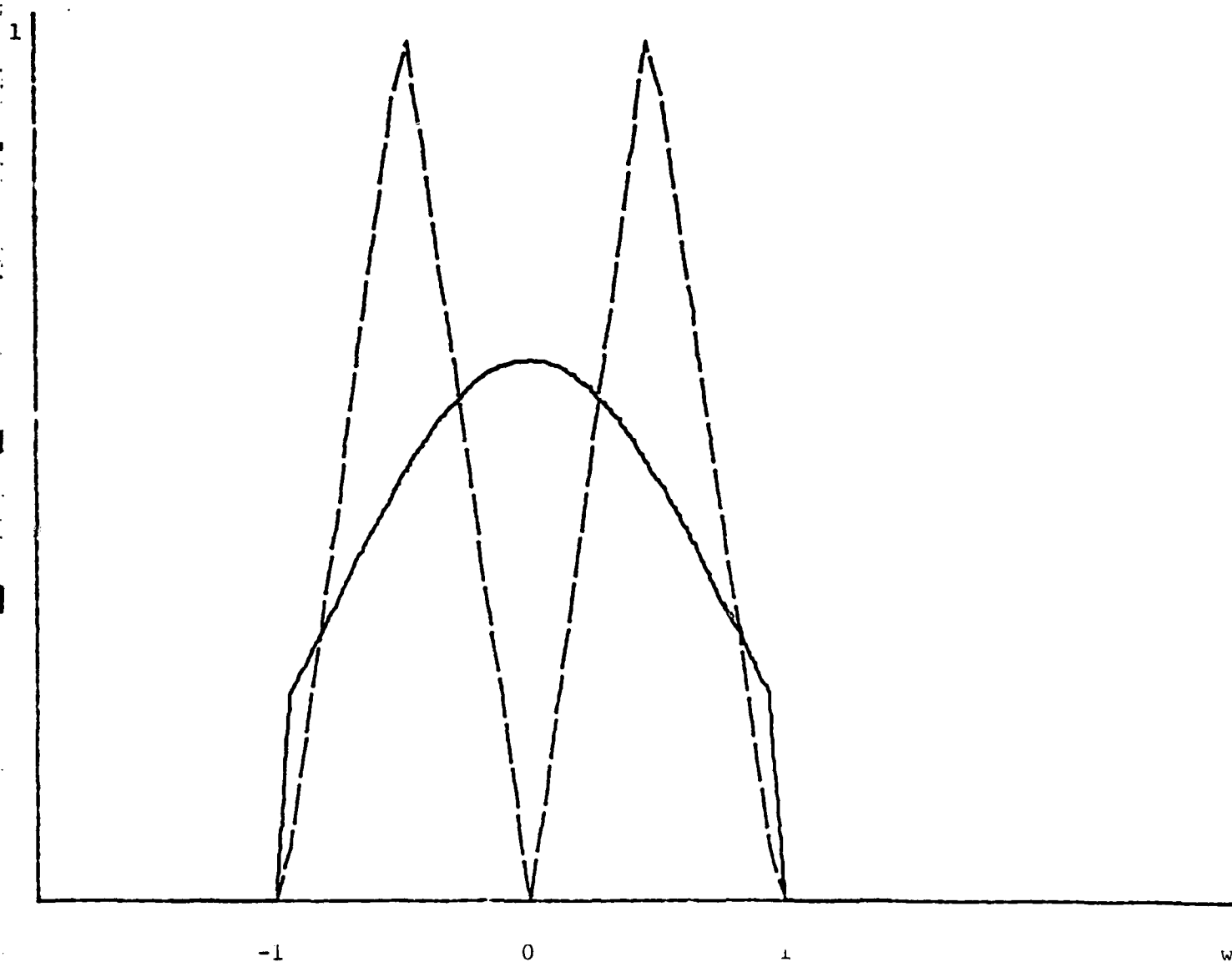
Figure 4a. Papoulis-Gerchberg's algorithm. 20 iterations.
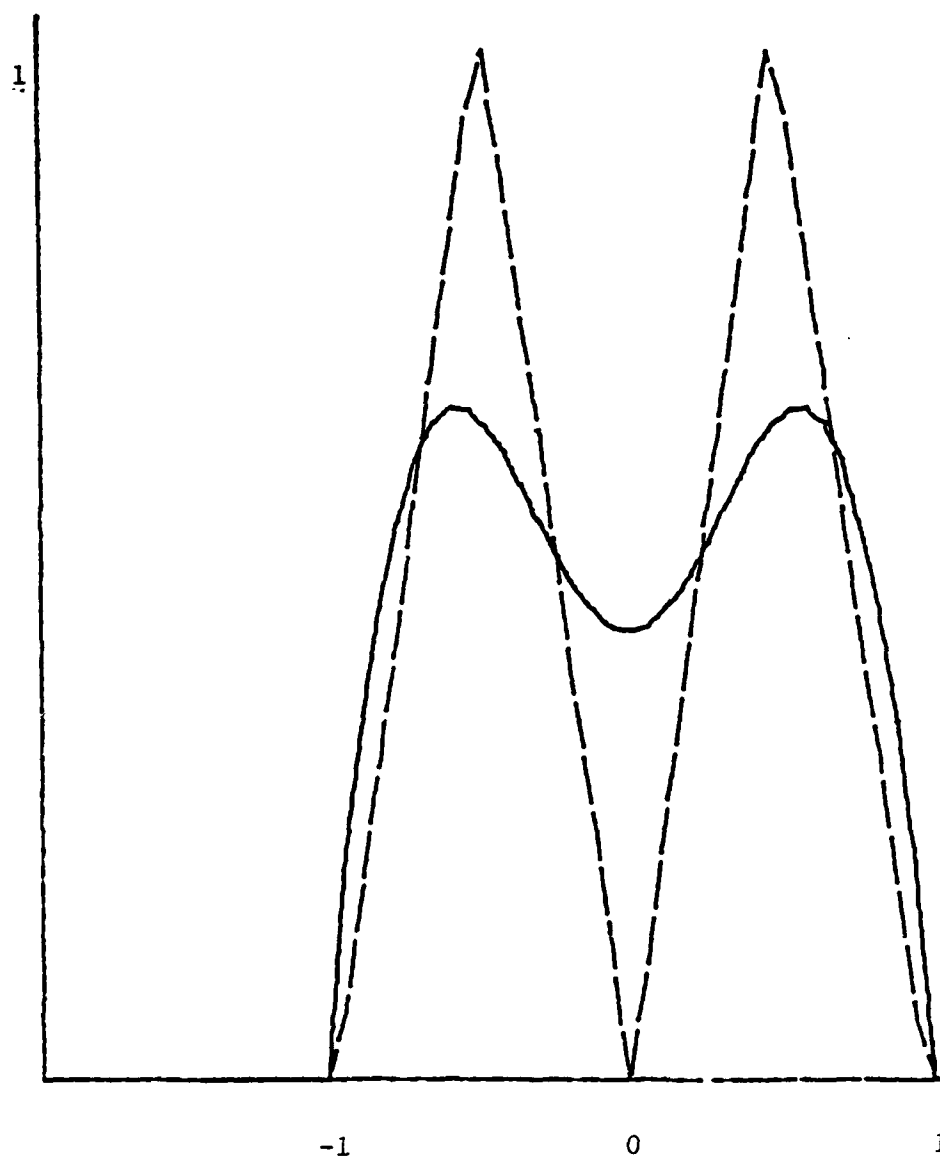
Figure 4b. Papoulis-Gerchberg's algorithm. 500 iterations.
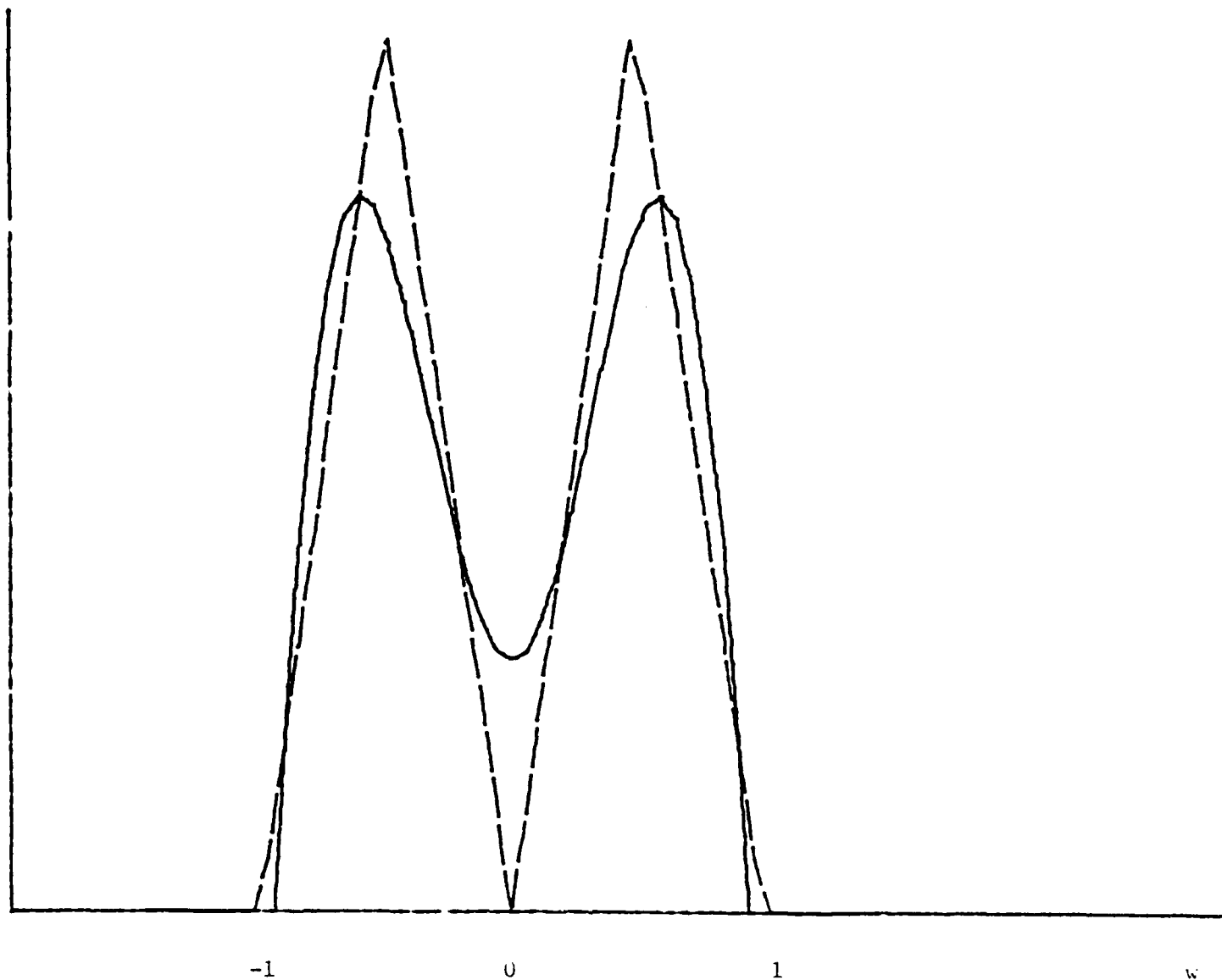
Figure 5a. $D = (A^t A + 0.00005I)^{-1}$.   10 iterations.
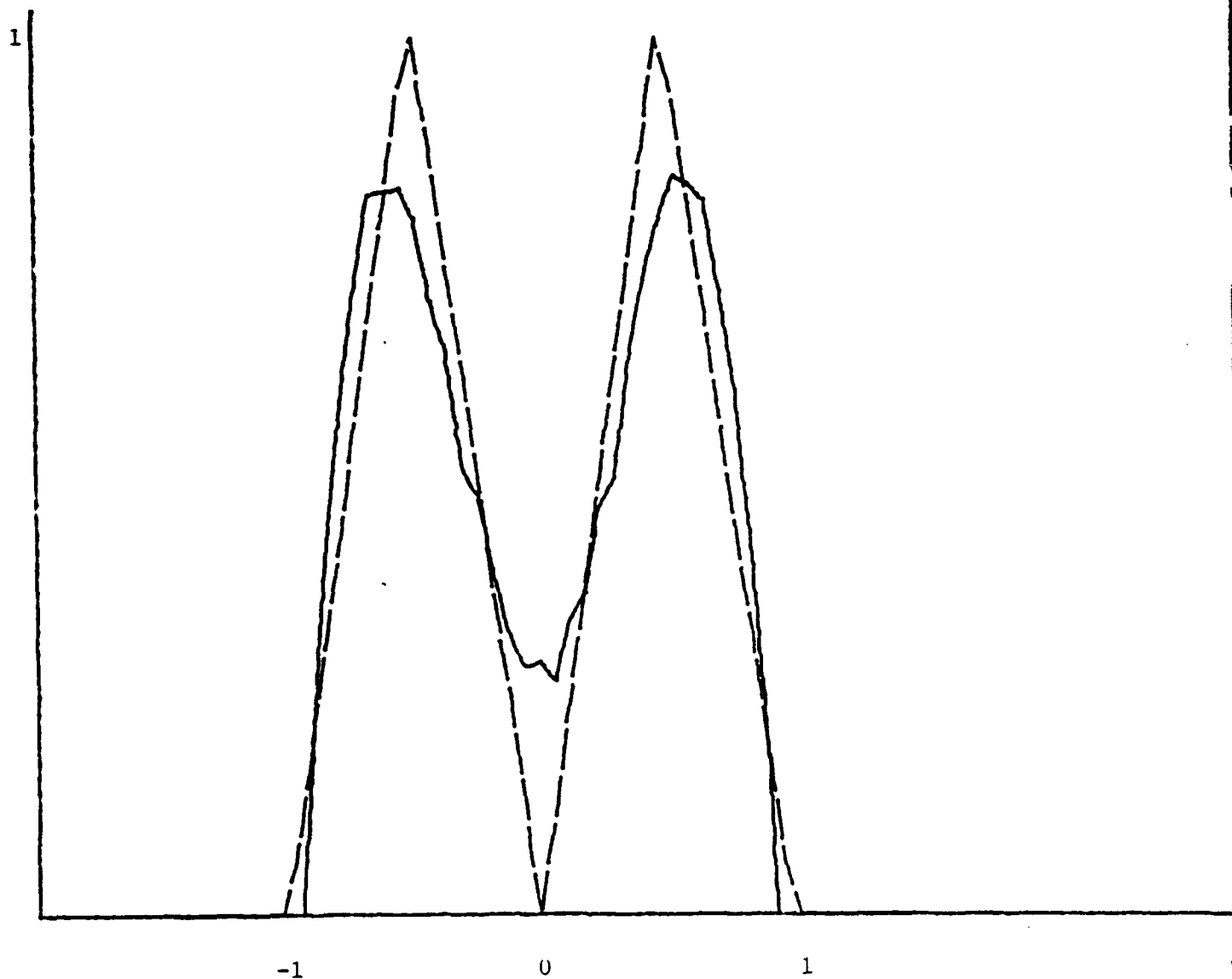
Figure 5b. $D = (A^t A + 0.00005I)^{-1}$. 20 iterations.
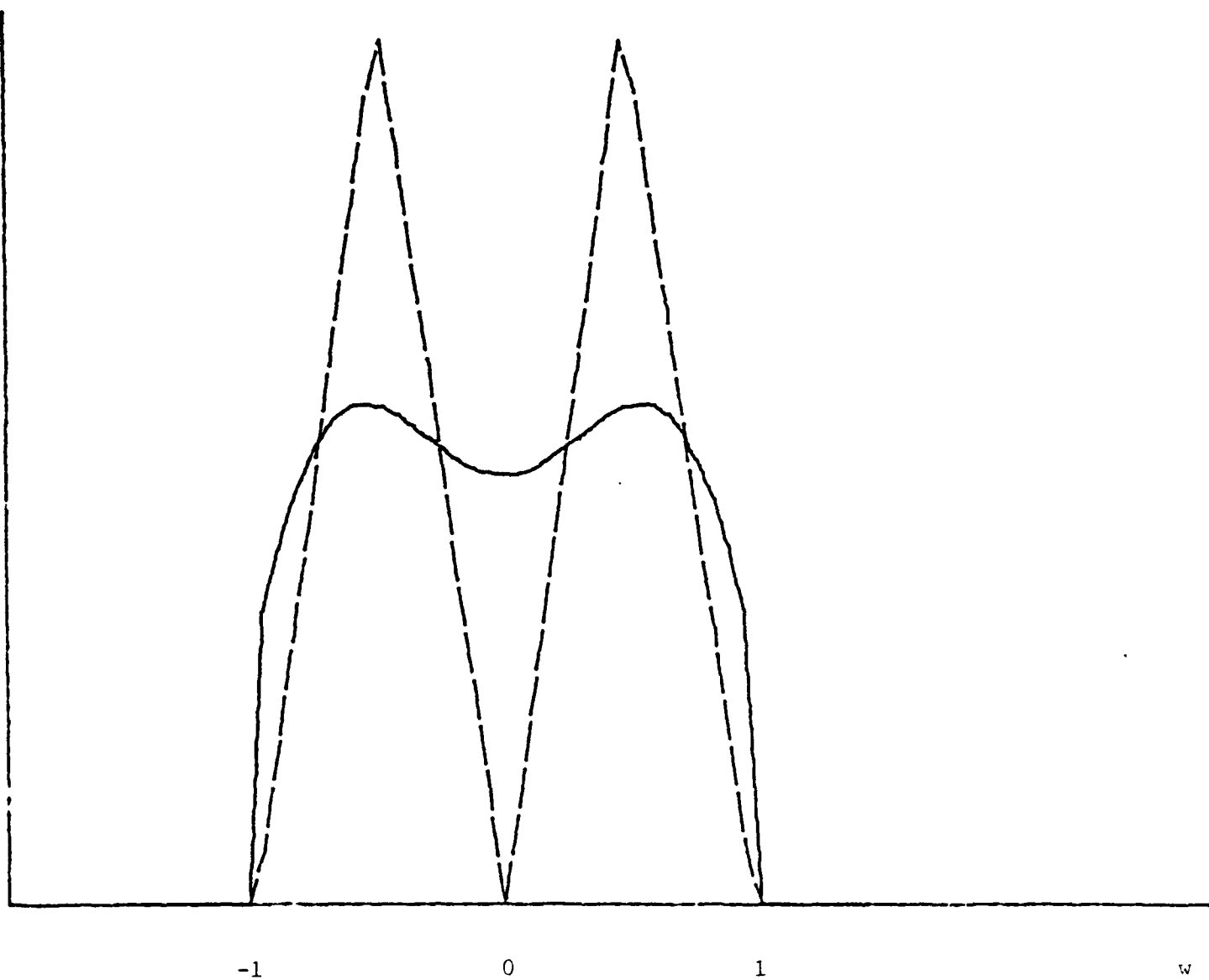
Figure 6a. $D = (A^t A + 0.005I)^{-1}$.   10 iterations.
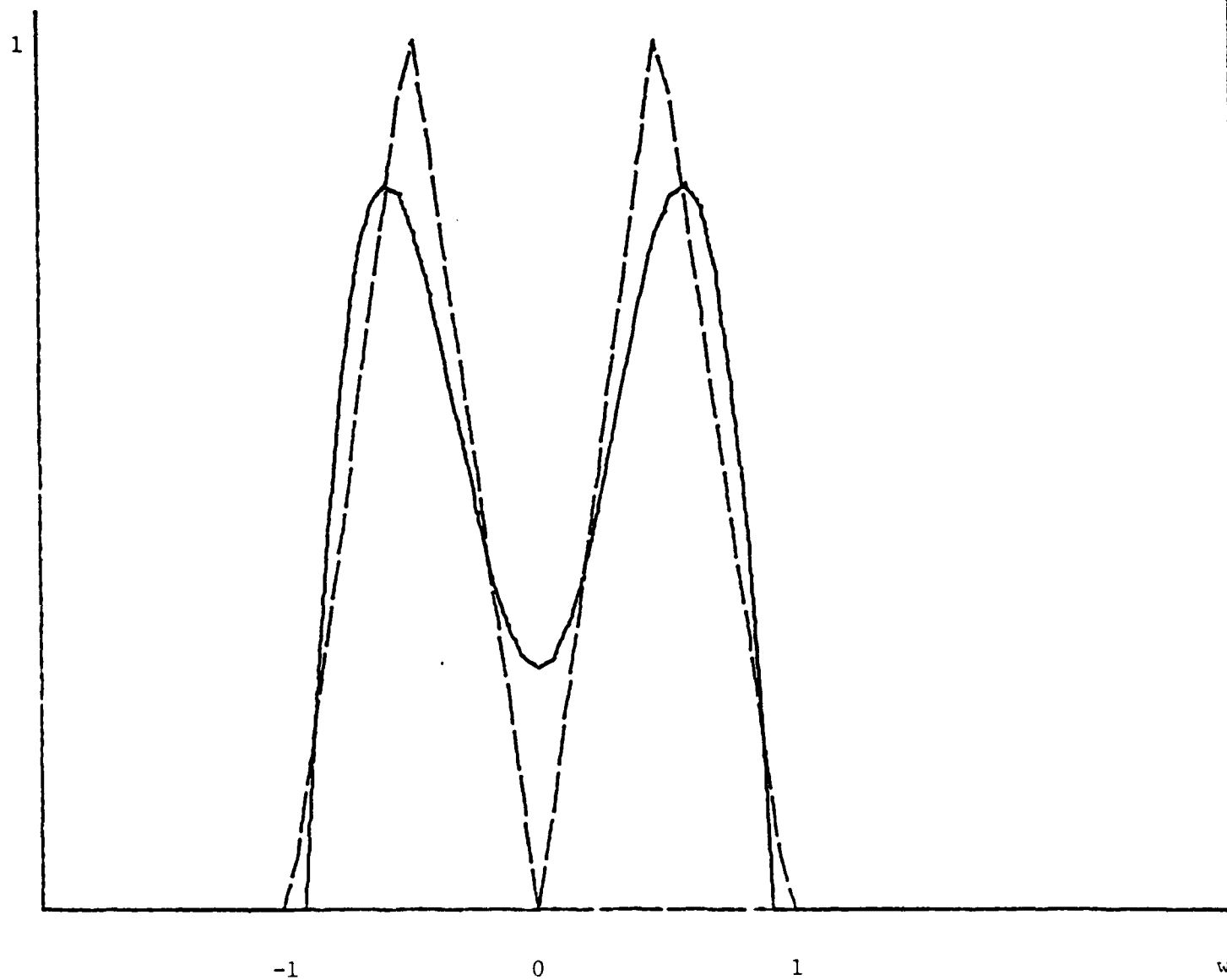
Figure 6b. $(\Lambda^t\Lambda + 0.005I)^{-1}$. 50 iterations.
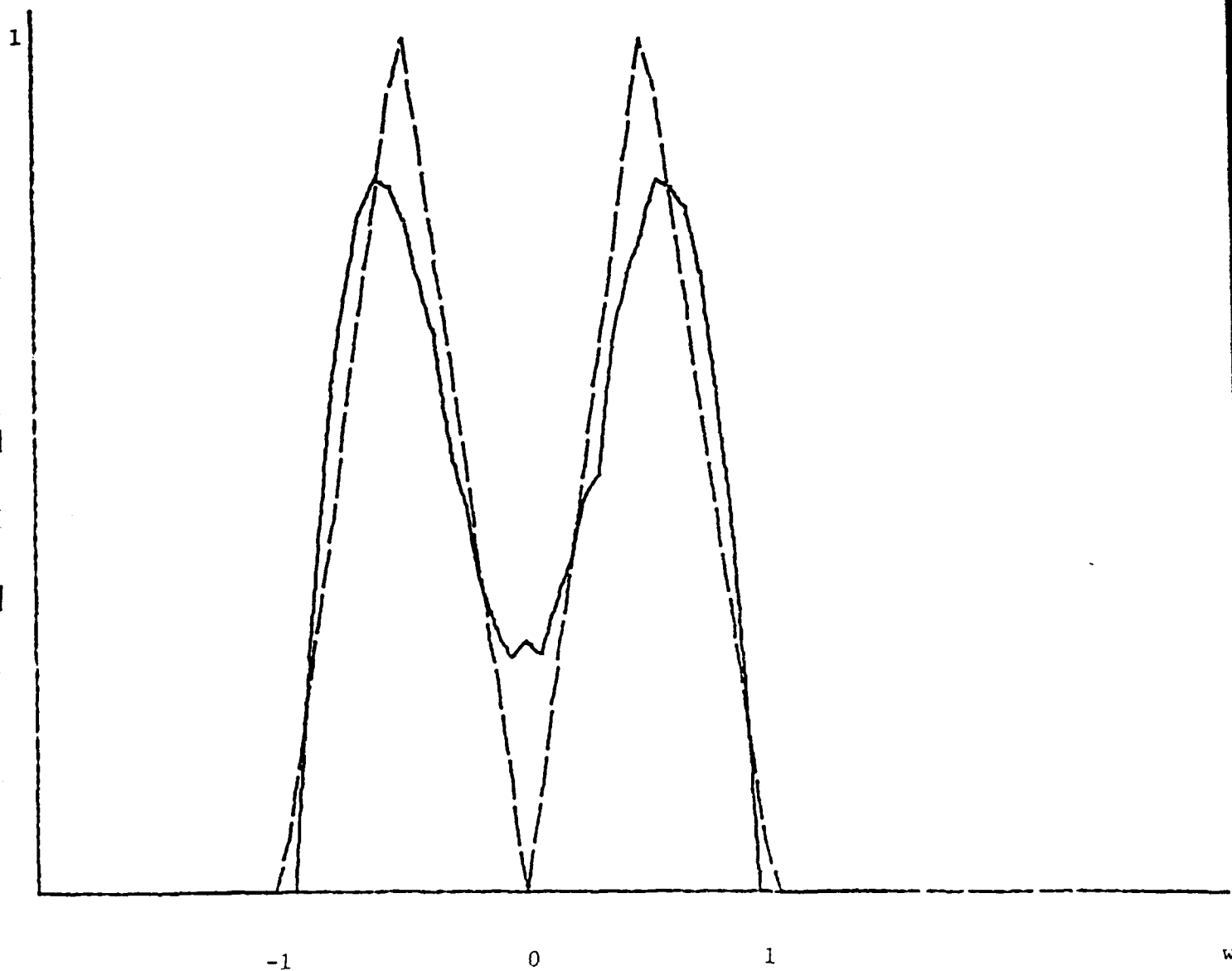
Figure 7. $D = (A^t A + 0.000005 I)^{-1}$. 10 iterations.
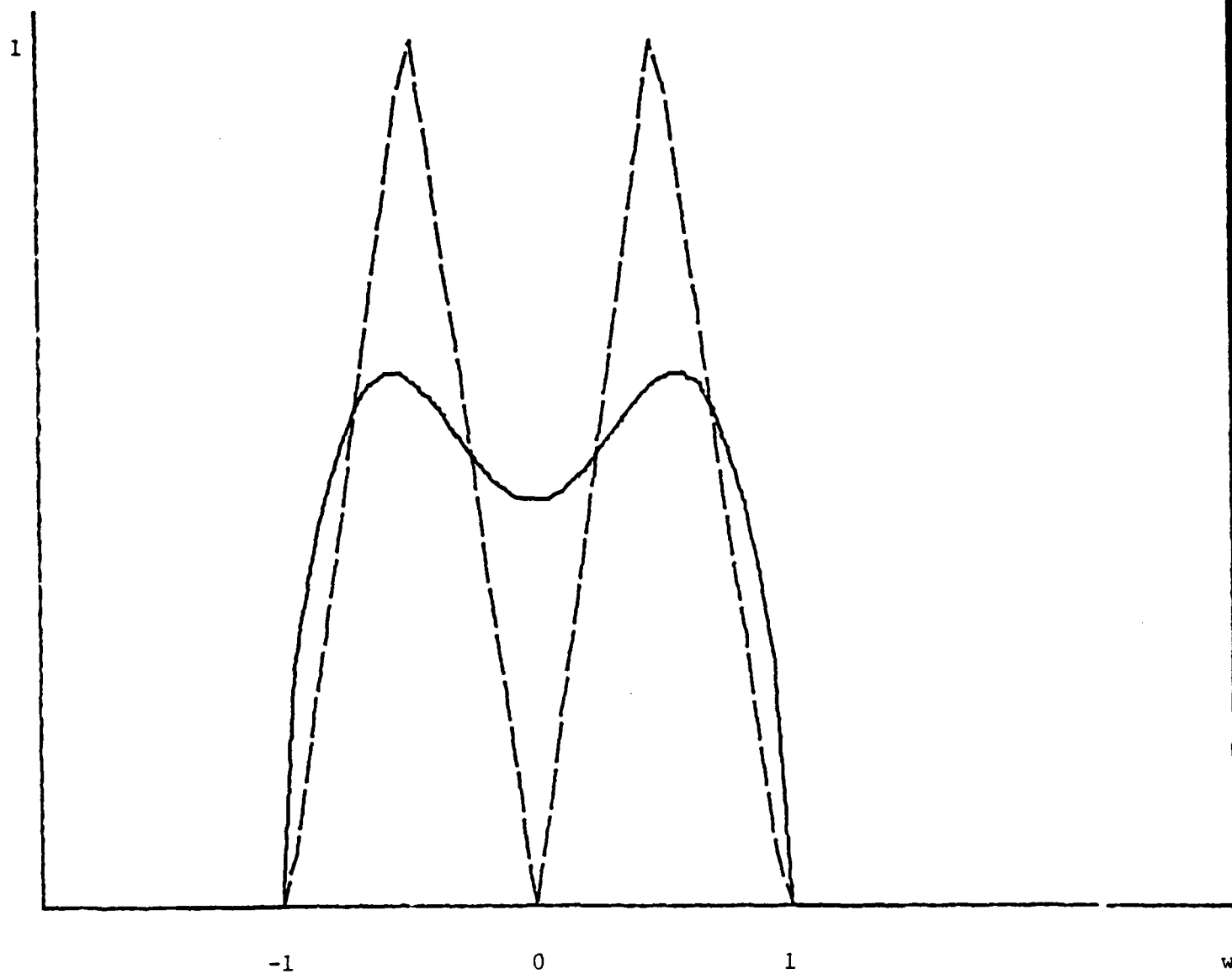
Figure 8a. $D = G(A^tA)$; G 6th order polynomial. 10 iterations.
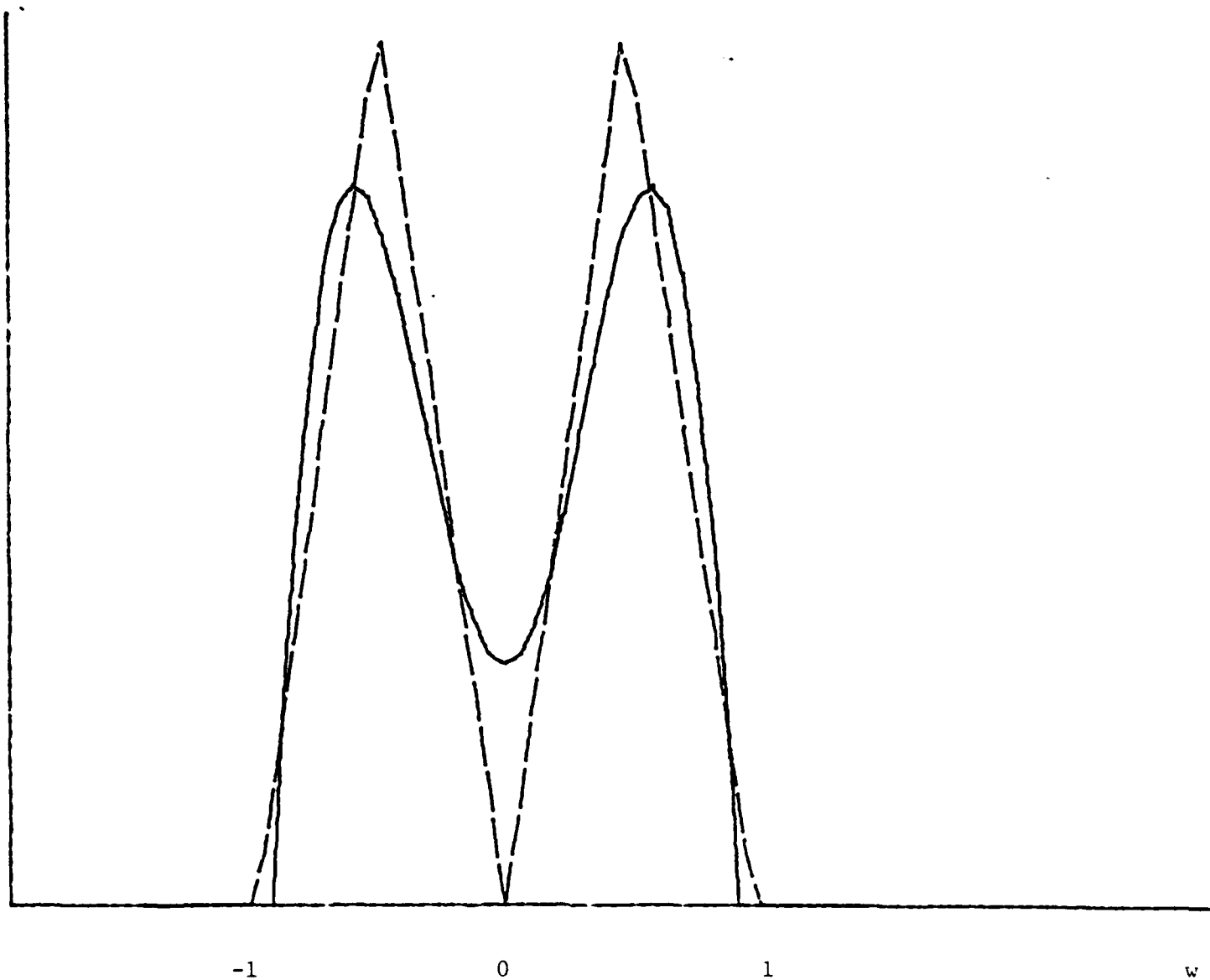
Figure 8b. $D = G(A^t A)$; G 6th order polynomial. 200 iterations.
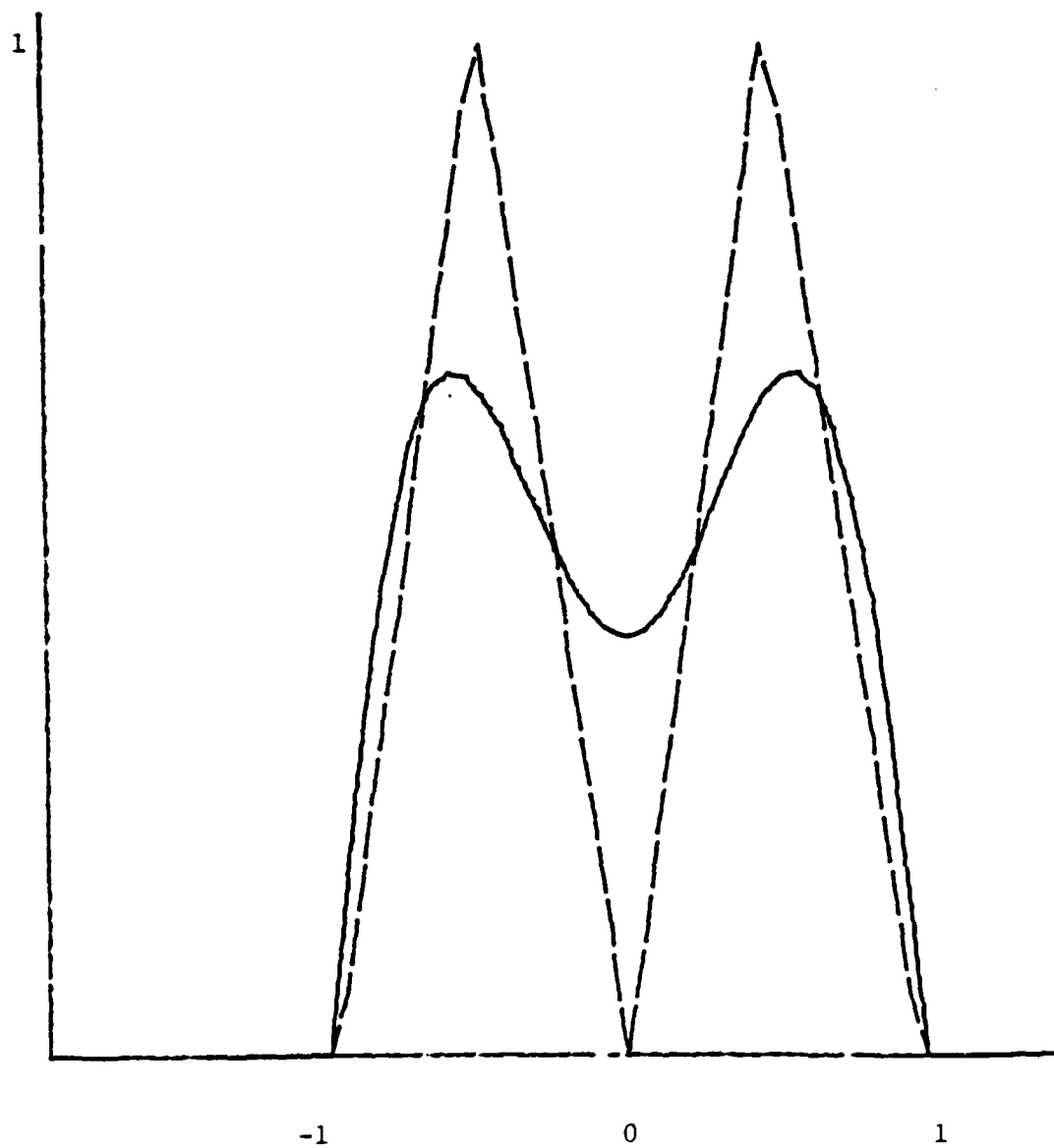
Figure 9a. $D = (\Lambda^t \Lambda)^{16}$.   500 iterations.
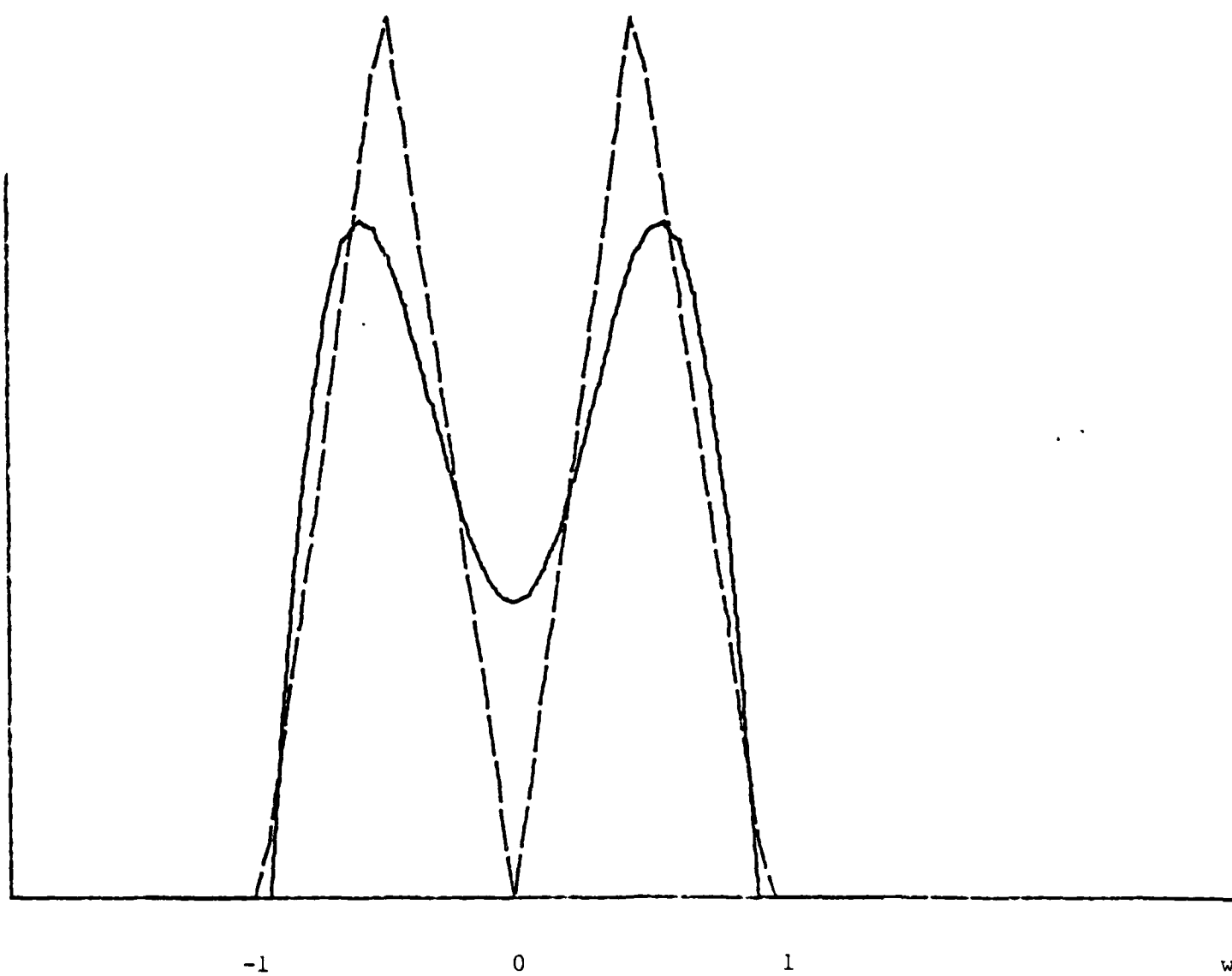
Figure 9b. $D = (\Lambda^t \Lambda)^{16}$. 1000 iterations.

CONTINUATION TECHNIQUES FOR A

CERTAIN CLASS OF ANALYTIC FUNCTIONS

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ABSTRACT

This paper deals with the numerical continuation problem of analytic functions $g(z)$ given by $g(z) = \int_{\Omega} K(z,t) \, f(t) \, dt$ where K defines a bounded operator on $L^2(\Omega)$. It is assumed that g is known over a finite segment A of the real line where g is to be sampled. Our continuation techniques emerged partly from a generalization of the Landweber iteration. We show that a certain discrete approximation of the proposed iterative technique yields two-step (non-iterative) algorithms for solving the continuation problem. We also prove convergence of these approximations to the sought function g. Special emphasis on the continuation problem for the case $K(z,t) = e^{-2\pi i z t}$ is given and some related numerical examples are presented. The continuation problem when the known part of $g(z)$ is contaminated with some noise is addressed and some techniques for solving this problem are also provided.

I. INTRODUCTION

Let us suppose that we are given a piece of a n-dimensional signal $g(t)$, $t \in A \subseteq \mathbb{R}^n$. In addition, we assume that g is obtained from some other signal $f(x)$, $x \in \Omega$, through a linear space variant system,

$$g(t) = \int_{\Omega} K(t,x) \, f(x) \, dx , \qquad t \in \mathbb{R}^n , \qquad (1)$$

where K is known. The goal is to recover the "real" object $f(x)$, $x \in \Omega$, from a finite set of samples of the "observed" signal $g(t)$, when $t \in A$. This problem is very well known in the engineering literature ([1]) and it has been extensively studied in the mathematical literature ([2]).

A very important case is obtained from (1) when $K(t,x) = k(t-x)$, $t,x \in \mathbb{R}^n$. In that case, g and f will satisfy the following relationship:

$$\hat{g}(\omega) = \hat{k}(\omega) \cdot \hat{Tf}(\omega) , \qquad \omega \in \mathbb{R}^n$$

where $\wedge$ denotes the Fourier transform and $TF(x) = f(x)$ if $x \in \Omega$, and 0 elsewhere. It is clear that

$$\frac{\hat{g}(\omega)}{\hat{k}(\omega)} = \hat{TF}(\omega) , \quad \text{for all } \omega : \hat{k}(\omega) \neq 0$$

Let us assume that $\hat{k}(\omega) \neq 0$ for $\omega \in N$, where N contains a non-void open set of $\mathbb{R}^n$. Since TF has compact support (if Ω is compact) then $\hat{TF}$ is analytic; so the knowledge of $\hat{TF}(\omega)$, when $\omega \in N$, will be enough to determine $\hat{TF}(\omega)$ for any other $\omega \in \mathbb{R}^n$. In many applications, $\hat{TF}(\omega)$, $\omega \in \mathbb{R}^n$ will describe by itself all the information that we need from TF. If this is not the case, then we should proceed to compute Tf from $\hat{Tf}(\omega)$, $\omega \in \mathbb{R}^n$. However,

we have assumed, so far, that $\hat{g}(\omega)$ is known exactly for all $\omega \in \mathbf{R}^n$. This will not be the case if $g(t)$ is observed on the set $A \neq \mathbf{R}^n$, or on a finite subset of A only. This shows that if we can improve our knowledge of g (i.e., to know $g(t)$, when $t \notin A$) we will obtain a better knowledge of $\hat{g}$ (i.e., to compute $\hat{g}(\omega)$ more accurately).

In many cases, $\hat{k}(\omega) = 0$ if $\omega \notin N$, where N is assumed to be compact. Therefore $\hat{g}(\omega) = 0$, $\omega \notin N$, which assures that g will also be an analytic function. This means that the set of values $g(t)$, $t \in A$ will determine $g(t)$, $t \notin A$. This shows that the solution f to the equation

$$g(z) = \int_\Omega k(z-x) \, f(x) \, dx$$

can be approached by solving a continuation problem for two analytic functions: $g(z)$, given $z \in A$, and $\hat{TF}(\omega)$, given $\omega \in N$. It is important to notice that the continuation of $\hat{TF}$ can be stated in the sense of equation (1) since

$$\hat{TF}(\omega) = \int_\Omega e^{-2\pi i x \omega} f(x) \, dx$$

and $e^{-2\pi i x \omega}$ plays the role of $K(\omega,x)$. This latter continuation problem shows an example of the importance of including space-variant kernels in our discussion.

One motivation for the continuation problem we give above is the restoration of f. However, there is another motivation: In many cases, we are interested in obtaining knowledge of $g(x)$, when $x \notin A$, and x is "close" to A. Some examples of this situation are known, in multidimensional signal

processing. Let us suppose we apply a filter to a given image. The filter
ideally performs over an infinite extent image. However, in practice we are
given only a piece of the image. When the filter is applied to points close
to or on the border of the real image, inaccuracies will result, if we assume
some arbitrary numbers for the unknown values of the image (e.g., the image
is assumed to be periodic; or a constant number is assumed for the unknown
values). The filter would improve its performance if we could fill out the
unknown values of the image with some interpolated information. Thus,
small amounts of extrapolation (i.e., to extrapolate a small region beyond
the boundary of A) can be of great help.

In the following sections we concern ourselves with the continuation
of the function g, when g is given by the equation

$$g(z) \ = \ \int_{\Omega} K(z,t) \ f(t) \ dt, \quad z \in \mathbb{R}^{n} \tag{1}$$

and $g(z)$ is known for $z \in A \subseteq \mathbb{R}^{n}$.

One possible approach for getting a continuation of g is to solve
equation (1) in terms of f and to use the same formula (1) to obtain the
continuation of g. We will use the Landweber-Strand ([3],[4]) iterative
procedure to get the solution of equation (1):

$$f_{o} \ = \ \text{initial approximation,}$$

$$f_{n} \ = \ f_{n-1} + D \ K^{*}(g - Kf_{n-1}) \ . \tag{2}$$

where K^{*} denotes the adjoint of K and D is a certain operator ([4]). In
section II, we will extend (2) to cases where K is not compact and D is any

suitable positive operator, including Strand's iteration ([4]) as a special case. If we apply K to both sides of equation (2) we will get

$$g_n = g_{n-1} + K D K^*(g - g_{n-1}) \qquad (3)$$

$$g_o = \text{initial guess} ,$$

where $g_n = Kf_n$ is a function defined on the whole R^n.

Under rather general conditions for K and f, recursion (2) is known to be convergent. On the other hand, we will also prove that the sequence g_n approaches g uniformly over compact sets in C^n. Practical computation of g_n requires equation (3) to be sampled; this means that discretization of g_n, g and KDK^* are unavoidable. In this paper, we will show that certain natural discretization provides an iterative procedure which is also convergent to a sequence. This sequence can also be obtained by solving a system of equations where D plays an essential role. This system of equations provides a natural interpolation for the sequence by means of an analytic function. We will also prove that this analytic function approaches g uniformly over compact sets when the distance between samples of g on A tends to zero. Thus, we will obtain a reliable technique for continuating g(z), because the closer the samples of g are taken on A, the better the approximation to g(z), when $z \notin A$. It turns out that our continuation technique does not require the computation of the derivatives of g. Section IV which contains the main results is derived independently of Landweber-Strand's iteration. However, Landweber-Strand's iteration is the origin of the main ideas developed in this paper and therefore, section II is fully devoted to this iterative procedure and our generalization. Section III shows the relationships between

Landweber-Strand's iteration and our continuation techniques, which are presented in Section IV. Section V includes some numerical examples for the case $K(x,y) = e^{-2\pi ixy}$. In section VI we present the continuation problem for the case in which the observations are given by $\tilde{g}(x) = g(x) + \eta(x)$, $x \in A$. Here, η denotes a continuous perturbation such that $|\eta(x)| \le \epsilon$ for some $\epsilon > 0$. All our results are presented for the one-dimensional case. However, extensions to the multidimensional case readily follow.

II. LANDWEBER-STRAND'S ITERATION

In this section we present Strand's iteration in a more general setting than that of [4]. Let $K : L^2(\Omega) \to L^2(A)$ be a bounded linear operator. Let $g \in L^2(A)$ be a function such that $g = g_1 + g_2$, where $g_1 \in R(K)$ and $g_2 \in R(K)^{\perp}$ ($R(K)$ denotes the range of K). In addition, let $D : R(K^*) \to R(K^*)$ be a bounded linear symmetric operator which is assumed to be one-to-one ($\forall x : Dx = 0$, then $x = 0$), and such that DK^*K is non-negative.

Theorem 1

Under the conditions stated above, let f_n be defined as follows

$$f_n = f_{n-1} + \alpha DK^*(g - Kf_{n-1})$$

$$f_o = \text{any initial guess} \in L^2(\Omega) \tag{4}$$

Let us suppose that $0 < \alpha < \frac{2}{\|DK^*K\|}$. Then, f_n converges to $(I - P)f_o + f^+$, where f^+ is the minimum norm least squares solution of $Kf = g$, P is the orthogonal projection on $N(K)^{\perp} = \left\{ v \in L^2(\Omega) : Kv = 0 \right\}^{\perp}$ and I denotes the identity.

Proof

This theorem is a simple consequence of a general result (Bialy [5]) for bounded operators on Hilbert spaces. Let us recall that our assumption $g \in R(K) + R(K)^{\perp}$ ensures that the problem $Kf = g$ has a least squares solution ([6]). It is also known that the set of least squares solutions is $\{x : K^*Kx = K^*g\}$. This means that we need to pick up the minimum norm solution for the operator equation

$$K^*Kx = K^*g \tag{5}$$

Since D is supposed to be one-to-one, equation (5) is equivalent to

$$DK^*Kx = DK^*g = g' \tag{6}$$

Since $DK^*K = H$ is a symmetric bounded linear and non-negative operator, Bialy's conditions [5] are satisfied and therefore the iteration

$$f_n = g_{n-1} + \alpha(g' - Hf_{n-1}) \tag{7}$$

$$f_o = \text{any initial approximation}$$

converges in the $L^2(\Omega)$ norm to $(I - P)f_o + f^+$, where f^+ is the minimum norm solution for the consistent equation $Hx = g$, provided that $0 < \alpha < \frac{2}{\|H\|}$ ([5]). It is also clear that (7) becomes

$$f_n = f_{n-1} + DK^*(g - Kf_{n-1})$$

We can include the relaxation parameter into D. In that case, convergence of (4) is ensured if $\|DK^*K\| < 2$. Strand ([4]) considers equation (4) for some special D and assuming K to be a completely continuous operator. If

the latter condition is satisfied then the singular value theory for compact operators will apply to our problem. In that case, if ϕ_n denotes the family of eigenfunctions of K^*K, for n : 1, 2, ..., and λ_n its corresponding eigenvalues:

$$K^*K\phi_n = \lambda_n\phi_n, \quad n = 1, 2, \ldots$$

Strand's conditions on D become

$$D\phi_n = p_n\phi_n, \quad n = 1, 2, \ldots \tag{8a}$$

where p_n satisfies

$$0 < p_n\lambda_n < 2 \text{ for all } n, \tag{8b}$$

$$p_n, \quad n = 1, 2, \ldots \quad \text{is bounded} \tag{8c}$$

It turns out that condition (8a) ensures that DK^*K is non-negative. It is also clear that (8b) is equivalent to $\|DK^*K\| < 2$, and condition (8c) ensures that D is a bounded operator on $\overline{R(K^*K)}$. These conditions allow for a partial study relating D to the behavior of the approximation (4) after n iterations.

Under the more general situation stated in theorem 1, some rational expressions in K^*K may play the role of D. Similar analyses to those of [4] are expected for this situation. However, this goes beyond the scope of this paper. In what follows, we will restrict our attention to cases in which K defines a bounded integral operator.

III. APPLICATIONS TO THE CONTINUATION PROBLEM

We remarked in the introduction that the iterative procedure (2) can be used for getting the continuation of certain analytic functions given by the formula

$$g(z) = \int_\Omega K(z,t) \, f(t) \, dt , \quad z \in \mathbb{R}$$

where K defines a bounded operator: $L^2(\Omega) \to L^2(A)$ and g is observed on the set A. It is clear that if we define $g_n = Kf_n$, where f_n is given by (2), we will obtain the recursion

$$g_n = g_{n-1} + KDK^*(g - g_{n-1})$$

$$g_0 = 0 = Kf_0, \quad f_0 = 0 \tag{9}$$

Since f_n converges to f^+ where f^+ is the minimum norm solution of the equation

$$g(z) = \int_\Omega K(z,t) \, f(t) , \quad z \in A ,$$

and since the convergence is in $L^2(\Omega)$, then g_n converges to g in $L^2(A)$. We now state the following result:

Corollary

Let us suppose that K satisfies these additional properties:

(1) $\int_\Omega K(z,t) \, h(t) \, dt$ defines an analytic function for $z \in C$ and any $h \in L^2(\Omega)$.

(2) $\sup\limits_{z \in \Gamma} \int_\Omega |K(z,t)|^2 \, dt \leq C_\Gamma < \infty$ for all compact sets $\Gamma \subseteq C$.

Then, the sequence g_n given by (9) converges uniformly over compact sets in

C to the continuation of $g(x)$, $x \in A$ for complex arguments $z \in C$.

Proof

Since $g_n = Kf_n$ is also an analytic function over the complex plane,

and $f_n \to f$ $L^2(\Omega)$ we have by means of the Schwartz inequality:

$$|(g - g_n)(z)| \leq \|f_n - f\|_2 \cdot \left[\int_\Omega |K(z,t)|^2 \, dt\right]^{\frac{1}{2}} \leq c_\Gamma \in$$

for $\in$ arbitrarily small, $n \geq n_o(\in)$ and Γ a compact set in C.

Some particular operators allow a bound (as the type of that given by

(2) in the Corollary) which depends only on

$$\sup_{z \in \Gamma} |Imz| \ .$$

In that case, convergence is assured to be uniform over the real line. This

is the situation for $K(z,t) = e^{-2\pi izt}$. Some consequences of this Corollary

in signal processing are given in [7].

Realistic applications of this iterative continuation formula involve

discretization of g, g_n and the operator KDK^*. Applying naive quadratures

formulas for the integral $K^*(g - g_{n-1})$ we will obtain the following recursion:

for any $k \in Z$:

$$S_n(k\Delta) = S_{n-1}(k\Delta) + \alpha \Delta \sum_{i=-N}^{N} \left[KD\overline{K}(i\Delta, \cdot)\right](k\Delta) \cdot (g(i\Delta) - S_{n-1}(i\Delta)) \quad (10)$$

$\overline{K}(x,y)$ is the complex conjugate of $K(x,y)$ and the dot in $\overline{K}(i\Delta, \cdot)$ indicates

the variable with respect to which D is being applied on $\overline{K}$; Δ is a positive number used to sample $2N + 1$ times the function g on $(-a,a)$: $\Delta = \frac{2a}{2N+1}$ and α is a relaxation factor independent of n. It would be desirable that $S_n(k\Delta) = g_n(k\Delta)$; however it is obvious that the sequence $S_n(k\Delta)$, $k \in Z$ is not obtained by sampling g_n. Two problems arise from the discrete recursion (10). The first is related to the convergence of the sequence S_n when $n \to \infty$. The second problem is about the relationship between this discrete technique and the original continuation problem. More specifically, the quewtion is whether

$$\lim_{n \to \infty} S_n(k\Delta)$$

will approach g when $\Delta \to 0$. Both problems are addressed in the next section.

IV. ALGORITHMS FOR THE CONTINUATION PROBLEM

In what follows, we will assume that $K(x,\cdot) \in L^2(\Omega)$ for all $x \in A$, $D = D^*$ is positive definite, and defined over the whole $L^2(A)$. Under these conditions the discrete recursion (10) will be derived by means of a rather different approach. Let us consider the following matrix L

$$L : \ell_{ij}, \quad -N \leq i,j \leq N.$$

$$\ell_{ij} = \Delta (KD \overline{K}(i\Delta,\cdot))(j\Delta) \tag{11}$$

It is simple to prove that L is a hermitian matrix, i.e. $\ell_{ij} = \overline{\ell_{ji}}$ for all i,j. If we assume that K has the following additional property:

$$\sum_{i=-N}^{N} c_i K(i\Delta,x) = 0, \text{ for all } x \in \Omega \quad - \quad c_i = 0, \quad i = -N, \ldots, N$$

$$\tag{12}$$

then L will be positive definite (without condition (12) L is non-negative). This shows that the system of equations

$$Lc = d \; ,$$

will always have a unique solution for any $d \in C^{2N+1}$. In particular, we can find γ_i, $i = -N, \ldots, N$ that satisfy

$$\Delta \sum_{j=-N}^{N} (KD \, \bar{K}(j\Delta,\cdot) \,)(i\Delta) \, \gamma_j = (Kf)(i\Delta), \quad -N \leq i \leq N \tag{13}$$

It is also worth pointing out that after γ_j is found in (13) we will have a natural interpolation formula for $g(i\Delta)$, by means of an analytic function, i.e.,

$$g_\Delta(z) = \Delta \sum_{j=-N}^{N} (KD \, \bar{K}(j\Delta,\cdot) \,)(z) \, \gamma_j \tag{14}$$

Let us show how this approach relates to recursion (10). Since L is a positive definite matrix, then γ_j, $-N \leq j \leq N$ can be obtained by means of the following well-known iterative procedure:

$$\gamma_i^{(n)} = \gamma_i^{(n-1)} + \alpha(g(i\Delta) - L\gamma^{(n-1)}) \tag{15}$$

α a certain positive relaxation parameter and

$$\gamma_i = \lim_{n \to \infty} \gamma_i^{(n)}, \text{ for all } i \in [-N,N].$$

We now apply $\Delta(KD \, \bar{K}(i\Delta,\cdot) \,)(k\Delta)$, $k \in \mathbb{Z}$, to both sides of formula (15) to get

$$\Delta \sum_{i=-N}^{N} (KD \, \bar{K}(i\Delta,\cdot) \,)(k\Delta) \cdot \gamma_i^{(n)} = \Delta \sum_{i=-N}^{N} (KD \, \bar{K}(i\Delta,\cdot) \,)(k\Delta) \cdot \gamma_i^{(n-1)}$$

$$+ \alpha \Delta \sum_{i=-N}^{N} (KD \, \bar{K}(i\Delta,\cdot) \,)(k\Delta) \cdot (g(i\Delta) - L\gamma_i^{(n-1)}) \tag{16}$$

If we call

$$S_n(k\Delta) = L\gamma_k^{(n)} = \Delta \sum_{i=-N}^{N} (K D \overline{K}(i\Delta, \cdot))(k\Delta)\gamma_i, \quad k \in \mathbb{Z},$$

equation (16) will become

$$S_n(k\Delta) = S_{n-1}(k\Delta) + \alpha\Delta \sum_{i=-N}^{N} (K D \overline{K}(i\Delta, \cdot))(k\Delta) \cdot (g(i\Delta) - S_{n-1}(i\Delta))$$

(10).

Since $\gamma^{(n)} \to \gamma$ when $n \to \infty$, then $S_n(k\Delta)$ converges when $n \to \infty$ and k is fixed. Moreover, we have proved that

$$\lim_{n \to \infty} S_n(k\Delta) = g_\Delta(k\Delta), \qquad \text{for all } k \in \mathbb{Z}.$$

We have shown, so far, that sampling iteration (g) produces a point-wise convergent sequence which can be computed by an exact technique. This procedure consists of two steps. We first solve a system of linear equations (13), obtaining γ_i, $i = -N, \ldots, N$; then, we use γ_i, to get an analytic function $g_\Delta(z)$, $z \in C$ which interpolates the set of samples $g(i\Delta)$, $i = -N, \ldots, N$, where $i\Delta \in A$ for all $i \in [-N,N]$. This function is given by equation (14):

$$g_\Delta(z) = \Delta \sum_{j=-N}^{N} (K D \overline{K}(j\Delta, \cdot))(z) \cdot \gamma_j$$

The remainder of this section is devoted to showing the relationship between $g_\Delta(z)$ and $g(z)$. We will need some lemmas.

Lemma 1

Let $(h_m)_m$ be a family of analytic functions over C^n and let us assume that $(h_m)_m$ is uniformly bounded over compact subsets of C^n (i.e., for all

compact sets $\Gamma \subseteq C^n$ there exists a constant c : $\sup_{z \in \Gamma} |h_m(z)| \leq c_\Gamma$ for all m).

Under these conditions there exists a sequence h_{m_k}, $k \to \infty$ and certain analytic function h such that h_{m_k} converges to h uniformly over compact subsets of C^n.

For a proof of this well-known result see, for instance, [8]. In what follows, we will show that g, $\frac{1}{.} = \frac{2a}{2N+1}$ satisfy the hypothesis of lemma 1, when D and f satisfy some additional conditions. We will also need another previous result from the optimization theory in Hilbert spaces.

Lemma 2

Let $(H, \langle,\rangle)$ be a Hilbert space. Let x_i, $-N \leq i \leq N$ be 2N+1 linear independent elements of H and c_i, $-N \leq i \leq N$ be 2N+1 complex numbers. Then, the following optimization problem:

$$\text{minimize } \langle x,x \rangle \tag{17a}$$

$$\text{subject to } x \in H$$

$$\langle x,x_i \rangle = c_i, \quad -N \leq i \leq N \tag{17b}$$

is uniquely solved by

$$x^o = \sum_{i=-N}^{N} \beta_i x_i \tag{18}$$

where β_i are determined by solving the system of equations:

$$\sum_{i=-N}^{N} \beta_i \langle x_i, x_j \rangle = c_j, \quad -N \leq j \leq N \tag{19}$$

For a proof of this lemma see, for instance, [9]. We are now able to prove

Theorem 2

Let us suppose that D satisfies: there exists $m > 0$: $\langle x, Dx \rangle \geqslant m \langle x, x \rangle$ for all $x \in L^2$, and assume that there exists a function $\ell \in L^2$ such that $D\ell = f$. In that case, the family of analytic functions g_Δ, $\Delta = \frac{2a}{2N+1}$ given by formulas (13) and (14) is uniformly bounded over compact subsets in C.

Proof

We will use lemma 2 where $x_i = \Delta^{\frac{1}{2}} \overline{K}(i\Delta, \cdot)$ and $c_i = g(i\Delta)$, $-N \leqslant i \leqslant N$. The underlying Hilbert space is $L^2(\Omega)$ equipped with

$$\langle x, y \rangle_s = \int_\Omega x(\omega) \overline{Dy(\omega)} \, d\omega .$$

Under these conditions, the optimization problem (17a) becomes

$$\text{minimize } \int_\Omega h(\omega) \overline{(Dh)(\omega)} \, d\omega \tag{20a}$$

$$\text{subject to } h \in L^2(\Omega)$$

$$\Delta^{\frac{1}{2}} \cdot \int_\Omega h(\omega) D\left[K(i\Delta, \cdot)\right](\omega) \, d\omega = g(i\Delta)$$
$$\text{for } -N \leqslant i \leqslant N \tag{20b}$$

It turns out that the solution to that problem is

$$\gamma^\circ(\omega) = \Delta^{\frac{1}{2}} \sum_{i=-N}^{N} \beta_i \overline{K}(i\Delta, \omega) , \tag{21}$$

where β_i satisfies

$$\Delta \cdot \sum_{j=-N}^{N} \beta_j \langle \overline{K}(j\Delta, \omega), \overline{K}(i\Delta, \omega) \rangle_s = g(i\Delta) , \quad -N \leqslant i \leqslant N \tag{22}$$

By definition of $< , >_g$, condition (22) becomes

$$\Delta \sum_{j=-N}^{N} \beta_j (KD \; \overline{K}(j\Delta,\cdot) \;)(i\Delta) \; = \; g(i\Delta) \; , \quad -N \leq i \leq N$$

This last system of equations coincides with (13) and therefore, $\beta_j = \gamma_j$, $-N \leq j \leq N$. It is also important to notice that g_Δ can be written as follows

$$g_\Delta(z) \; = \; \Delta^{\frac{1}{2}} \int_\Omega \dot{\gamma}^0(\omega) \; D \; K(z,\cdot) \; (\omega) \; d\omega \tag{23}$$

where γ^0 is the optimum given by equation (21). Since $D\ell = f$, for certain $\ell \in L^2$, the function $\Delta^{-\frac{1}{2}} \cdot \ell$ is a feasible point for the optimization problem, that is to say, $\Delta^{-\frac{1}{2}} \cdot \ell$ satisfies condition (20b). Hence,

$$\int_\Omega \gamma^0(\omega) \; D\gamma^0(\omega) \; d\omega \; \leq \; \Delta^{-1} \int_\Omega \ell(\omega) \; D \, \ell(\omega) \; d\omega \tag{24}$$

because γ^0 is the "minimun-norm" function among those which satisfy (20b). Now, the sought property of the family g_Δ readily follows from (23) and (24). From (23) we obtain

$$| g_\Delta(z) | \; \leq \; \Delta^{\frac{1}{2}} \cdot \|\gamma^0\|_2 \cdot \| D \; K(z,\cdot) \|_2 \tag{25}$$

Using equation (24), and the assumptions made on D and K, (25) becomes

$$| g_\Delta(z) | \; \leq \left(\int \ell(\omega) \; D\ell(\omega) \; d\omega \right)^{\frac{1}{2}} \|D\|_2 \cdot \left(\int |K(z,t)|^2 \; dt \right)^{\frac{1}{2}} \cdot \frac{1}{m^{\frac{1}{2}}} \tag{25'}$$

If we now assume that $z \in \Gamma$ a compact set in C, equation (25') becomes

$$| g_\Delta(z) | \; \leq \; c_\Gamma \quad \text{for all } z \in \Gamma \; ,$$

where c_Γ depends only on Γ.

We are now able to apply Lemma 1 to obtain

Theorem 3

Let

$$g_\Delta(z) = \Delta \sum_{j=-N}^{N} \gamma_j (KD \, \overline{K}(j\Delta, \cdot))(z) \, , \quad z \in C \, , \quad \Delta = \frac{2a}{2N+1}$$

be the analytic function obtained with the condition

$$-N \leqslant i \leqslant N : \quad \Delta \sum_{j=-N}^{N} \gamma_j (KD \, \overline{K}(j\Delta, \cdot))(i\Delta) = g(i\Delta) = (Kf)(i\Delta) \qquad (13)$$

Let us assume that D and K satisfy all the conditions stated in Theorem 2. In that case, there exists a subsequence $\Delta_m \rightarrow 0$ where $m \rightarrow \infty$ such that g_{Δ_m} approaches g uniformly over compact sets in C.

Proof

By lemma 1, there exists a subsequence $\Delta_m \rightarrow 0$ and an analytic function $h(z)$, such that g_{Δ_m} approaches h uniformly over compact sets in C. In addition, h = g because g is a family of equicontinuous functions (a simple consequence of the uniform boundness) and therefore, since $g_{\Delta_m}(i\Delta_m) = g(i\Delta_m)$ when $-[2a/\Delta_m] \leqslant i \leqslant [2a/\Delta_m]$ and the set of numbers $i\Delta_m : 0 \leqslant m < \infty$, $-[2a/\Delta_m] \leqslant i \leqslant [2a/\Delta_m]$ is dense in $(-a,a)$ we conclude $h(z) = g(z)$ for $z \in (-a,a)$. Since both functions are analytic, the same identity holds for $z \in C$.

We would like to remark on some important points and to give some examples before ending this section.

The property that DK^*K is positive was not used in the proof of Theorem 3. On the contrary, this property is required for the recursive computation of f by means of formula (2). It is worth pointing out that we have used, so far, equally spaced samples of $g(i\Delta)$ where $\Delta = \frac{2a}{2N+1}$. However, similar results to those of theorem 2 and theorem 3, and different formulas for interpolating samples (as (14)) can be obtained for non-equally spaced samples and for other regular distributions. It is also easy to see that we could have used the following sampling sets:

$$\left\{ i\Delta_{N_k} , \quad 0 < k < \infty; \quad -N_k \leq i \leq N_k; \quad \Delta_{N_k} = \frac{2a}{2N_k + 1} \right\}$$

where N_k is a non-decreasing infinite sequence of natural numbers. In that case, by means of theorem 3 we would obtain a subsequence of N_k, $k = 1, 2, \ldots$, say Θ_n, $n = 1, 2, \ldots$, such that g_{Θ_n} approaches g uniformly over compact sets in the complex plane. If we repeat this procedure for every non-decreasing infinite sequence of natural numbers we will conclude the convergence of g_{Δ_N} to g, where $\Delta_N = \frac{2a}{2N+1}$, and N is any natural number. This observation is important because it means that our approximation to g over a compact set, by means of g_{Δ_N} will be good for all $N \geq N_o$ if it is good for N_o. This observation also applies to more general sequences of sampling sets.

We now give some specific examples of operators which may play the role of D in the continuation technique. Probably the simplest case is D = I, the identity operator. In that case, the interpolation formula (14) becomes

$$g_\Delta(z) = \Delta \sum_{j=-N}^{N} K\left(\overline{K}(j\Delta, \cdot)\right) (z) \, \gamma_j \qquad (26)$$

where γ_j are chosen to satisfy $g_\Delta(i\Delta) = g(i\Delta)$, $i = -N, \ldots, N$. Another example is $D = (K^*K + \alpha I)^{-1}$, where α is a real positive number. More generally, if P_k denotes the polynomial

$$P_k(\lambda) = \sum_{i=0}^{k} a_i \lambda^i , \quad \text{where } a_i \geqslant 0 , \quad a_1 > 0$$

then $D = P_k(K^*K)$ will satisfy all the conditions required by theorem 3 and so does D^{-1}. The examples given above satisfy $R(D) = L^2(\Omega)$. This property ensures that $f \in R(D)$ for any $f \in L^2(\Omega)$ whatsoever. In particular if f is such that $Kf = g$, we will have $f \in R(D)$. Hence, no relationships between f and D are needed to apply theorem 3. Unfortunately, this is not always the case. The other property which D is required to satisfy is

$$\exists \; m > 0 : \; \langle Dx, x \rangle \geqslant m \langle x, x \rangle \text{ for all } x \in L^2(\Omega) \tag{27}$$

This assumption is by no means mild. Simple positive operators may not have this property. For example, let us define

$$D : L^2(\Omega) \to L^2(\Omega) : (D\ell)(\omega) = \hat{h}(\omega) . \ell(\omega) \tag{28}$$

where $\hat{h}$ is a bounded, real and positive function. Even if $f \in R(D)$, that is to say

$$\left\| \frac{f}{\hat{h}} \right\|_2 < \infty , \tag{28'}$$

D may not satisfy (27) and therefore, the hypotheses of theorem 3 may not be satisfied. However, the thesis of that theorem still holds. To show that, let us assume

$$\int_\Omega \frac{|f|^2}{\hat{h}} \, d\omega \tag{29}$$

which is, in fact, a weaker condition than (28). Then, the analytic function g can be written as

$$(KD^{\frac{1}{2}}) \frac{f}{h^{\frac{1}{2}}} = g , \quad \frac{f}{h^{\frac{1}{2}}} \in L^2(\Omega)$$

where $D^{\frac{1}{2}} : L^2(\Omega) \to L^2(\Omega) : (D^{\frac{1}{2}}\ell)(\omega) = h^{\frac{1}{2}}(\omega).\ell(\omega)$. We now call $H = KD^{\frac{1}{2}}$ and we apply theorem 3 when K is replaced by H (notice that H is an integral operator) and when no D is present (i.e., D = I). We obtain the interpolation (26):

$$g_\Delta(z) = \Delta \sum_{j=-N}^{N} H \bar{H}(j\Delta,\cdot) (z) \gamma_j$$

or equivalently

$$g_\Delta(z) = \Delta \sum_{j=-N}^{N} KD \bar{K}(j\Delta,\cdot) (z) \gamma_j \tag{30}$$

which is the formula we should obtain in theorem 3 applied to this situation.

The great flexibility in choosing D shown in theorem 3 may be useful in two different ways:

1. Some more a priori information about g (or f) may suggest picking up some D's to get closer interpolation formulas. The numerical evidence of the next section gives an example about this situation.

2. The noise problem. When g is contaminated with some noise, the effect of D may be of great importance, especially when the system of equations (13) is solved iteratively. This has been shown to be the case when we deal with the iterative procedure (2) for computing f (see [4]). Some numerical properties of related techniques for solving the problem in the presence of noise were recently reported in [13] when $K(x,t) = e^{-2\pi i x t}$.

To finish this section we would like to point out that another

interesting question remains unanswered: What are the additional conditions that f and D should satisfy to get an approximation result such as that of theorem 3 when D is any positive-definite linear operator?

IV. NUMERICAL EXAMPLES AND DISCUSSION

We will consider in this section some numerical examples for the case $K(z,t) = e^{-2\pi izt}$. The special interest in this particular operator is well known and we tried to explain some of the reasons in Section I.

In this situation, interpolation formula (14) becomes

$$g_\Delta(z) = \Delta \sum_{j=-N}^{N} \gamma_j \int_\Omega e^{-2\pi izx} D(e^{2\pi ixj\Delta}) dx \qquad (31)$$

The case D = I was addressed in [12]. However, in ref. [12] it was also assumed that $g \in R(K^*K)$ to derive the corresponding interpolation formula and no relationships between g_Δ and g were shown. On the contrary, we do not need any further restriction on g than $g \in R(K)$, that is to say,

$$g(z) = \int_\Omega f(t) e^{-2\pi izt} dt , \quad z \in A$$

and since the kernel satisfies all the properties required by theorem 3 we also know that $g_\Delta \to g$ uniformly over compact sets. In ref. [10] it was shown that for this particular situation the convergence of g_Δ to g is uniform over the whole real line.

If we assume that D is given by (28) and if $\int_\Omega \dfrac{|f|^2}{\hat{h}} d\omega < \infty$ where $\hat{h} : \mathbb{R} \to \mathbb{R}$ is bounded and $\hat{h}(\omega) = 0$ if $\omega \notin \Omega$, then we can apply formula (30) to obtain:

$$g_\Delta(z) = \Delta \sum_{j=-N}^{N} \gamma_j \, h(z-j\Delta) \tag{32}$$

In this situation, the system of equations we need to solve to compute γ_j, $j = -N, \ldots, N$ has the following form

$$g(i\Delta) = \Delta \sum_{j=-N}^{N} \gamma_j \, h((i-j)\Delta) \tag{33}$$

It turns out that the matrix involved in (33) is Toeplitz and therefore, $O((2N + 1)^2)$ algorithms can be used for solving the system. We have developed an iterative approach for solving (33) whose relaxation parameter does not depend on the number of points inside A (for Δ fixed), when A is varying ([11]).

Our numerical experience will be based on the following functions:

$$g_1(z) = \left(\frac{\sin \frac{\pi}{2} z}{\frac{\pi}{2} z} \right)^2 \cos \pi z \quad , \quad z \in C$$

$$g_2(z) = \frac{\sin 2\pi(z-\frac{1}{2})}{\pi(z-\frac{1}{2})} + \frac{\sin 2\pi(z+\frac{1}{2})}{\pi(z+\frac{1}{2})} \quad , \quad z \in C$$

$$g_3(z) = \left(\frac{\sin \pi z}{\pi z} \right)^2 , \quad z \in C$$

It is clear that these functions are analytic and they can be written as

$$\int_\Omega f(x) \, e^{-2\pi izx} \, dx$$

with $f \in L^2(\Omega)$ and $\Omega = (-1, 1)$. The interval $(-0.5, 0.5)$ is assumed as the

set of known values. We will take 33 equally spaced samples in (-0.5, 0.5) with $\Delta = \frac{1}{33}$, for each case. The continuation will be given by means of 65 equally spaced samples of $g_\Delta(z)$ taken in (-1, 1), corresponding to the points $z = i\Delta$, $i \in [-32, 32]$.

In the first case, $g_1(z)$ is given by the integral

$$\int_{-0.5}^{0.5} f_1(\omega) \ e^{-2\pi i \omega z} \ d\omega$$

where f_1 is drawn in figure 1.a. The set of 33 samples of g_1 is given in table 1.a. Figure 1.b shows the function g_1 in (-1,1). These samples provide a very poor description of the spectrum of $g(x)$, $x \in \mathbb{R}$, because their discrete Fourier transform (DFT) does not distinguish the corresponding pair of peaks. The DFT is shown in table 1.b and in figure 2.2. Compare this with the DFT obtained when the 33 samples are taken in (-1,1) (fig. 2.b). In this case we have chosen D = I and therefore, g is given by (32) with

$$h(x) \ = \ \frac{\sin 2\pi x}{\pi x} \ .$$

Table 1.c shows the values of the continuation $g_\Delta^{(1)}(i\Delta)$, $32 \geqslant |i| \geqslant 17$, $\Delta = \frac{1}{33}$. Table 1.d shows the real values $g_1(i\Delta)$, $32 \geqslant |i| \geqslant 17$. It is seen that the continuation is of good quality. We now compute the DFT of the sequence defined by the given samples $g_1(i\Delta)$, $0 \leqslant |i| \leqslant 16$ plus the continuation samples $g_\Delta^{(1)}(i\Delta)$, $32 \geqslant |i| \geqslant 17$. The result is given in table 1.3. Notice that the two peaks are now distinguished very clearly. This example shows the importance and effectiveness of the continuation procedure.

The second case is intended to be an example of the effectiveness of the algorithm when the main peaks of the function to be continuated are not completely included into the known range (fig. 3). In this example, we use

$D = K^*K + \varepsilon I$, where $\varepsilon = 0.5$. The function obtained by means of the formula $KD(\overline{K}(x,\cdot))(z)$ can be written in this case as

$$\int_{-0.5}^{0.5} \text{sinc}_{1.0}(z-y) \, \text{sinc}_{1.0}(y-x) \, dy + 0.5 \, \text{sinc}_{1.0}(z-x) \qquad (34)$$

where

$$\text{sinc}_{1.0}(\omega) = \frac{\sin 2\pi\omega}{\pi\omega} .$$

The function obtained by the continuation procedure is sampled, giving $g_{\Delta}^{(2)}(i\Delta)$, $32 \geqslant |i| \geqslant 17$ (table 2.b). The values of the known samples of g_2 are listed in table 2.a, and the real values $g_2(i\Delta)$, $32 \geqslant |i| \geqslant 17$ are given in table 2.c. The conclusion that can be drawn by comparing tables 2.b and 2.c is that the continuation procedure performs a fairly reasonable extrapolation. It is worth noticing the accuracy obtained at the main peaks. However, these peaks are still close to the known range of the given function. Further numerical evidence is necessary when the location of the peaks is more critical.

The third case is to provide a rough comparison of the performance of two different D's. The function g_3 (fig. 4) is to be continuated from its samples: $g_3(i\Delta)$, $0 \leq |i| \leq 16$ by means of $D_1 = \text{sinc}_{1.0}(x-y)$ and D_2 given by (34). Table 3.a shows the known samples. Tables 3.b and 3.c list the values of both continuations. Table 3.d contains the real values of g_3 taken on the same points at which the continuations are sampled. It is seen that the continuation provided by D_2 is closer to g_3 than that of D_1 on most of the sample points. This phenomenon is not yet fully explained.

Probably the main conclusion that may be drawn from these examples is

that 33 samples of the given function was always enough to get a quite reasonable continuation. It goes without saying that this result is due to some peculiar properties of the given functions g_1, g_2 and g_3, in terms of the chosen D's. A deeper study of these properties might shed more light on the role played by D in the continuation technique.

IV. CONTINUATION AND NOISE

We have assumed, so far, that our samples of g, taken on A, are not contaminated with any noise. However, a more realistic model should consider that the given observations are $\tilde{g}(x)$, $x \in A$, where $\tilde{g} = g + \eta$. It is clear that g and η cannot be separated. In general, η is not a smooth noise and therefore, we need to change the formulation of the continuation problem because g is not a piece of some analytic function. We will assume that $\eta : A \rightarrow C$ is a continuous function which satisfies $|\eta(x)| \leqslant \in$, for all $x \in A$, where $\in$ is some known constant. Under these assumptions, we could seek for a function h(z) which is given by $\int_{\Omega} K(z,t)\, s(t)\, dt$, for all $z \in C$, $s \in L^2(\Omega)$ and such that $|\tilde{g}(x) - h(x)| \leqslant \in$, for all $x \in A$. It is obvious that this problem has a solution. However it may not be unique and therefore, h will be different from g. This means that in the absence of any other a priori information about g, the continuation of g from $\tilde{g}$ becomes a delicate matter.

Among the infinitely many functions which may play the role of h (i.e., which satisfy the two conditions stated above), we may put some additional constraint to the problem to guarantee a unique solution: $h_\in$ for all $\in > 0$. The only restriction we should impose to the additional constraint (in the absence of some more a priori information about g) is that the

solution $h_\in$ approaches (in some convenient way) the true function g when $\in$ tends to zero. This convergence must involve the values of g beyond A. In this section, we develop a technique which will use the following three constraints:

$$h_\in(z) = \int_\Omega s_\in(t) \, K(t,z) \, dt \, , \quad z \in C \tag{35a}$$

$$|h_\in(x) - \widetilde{g}(x)| \leq \in \, , \quad \text{for all } x \in A \tag{35b}$$

$$s_\in \in L^2 \text{ minimizes } \langle s,s \rangle_D = \int_\Omega s(t) \, \overline{(Ds)}(t) \, dt \tag{35c}$$
$$\text{among those which satisfy (35a) and (35b)}$$

In formula (35c), D denotes a linear, bounded operator which satisfies $\langle Dx,x \rangle \geqslant m \langle x,x \rangle$ for all x, with m > 0. We will give a procedure to compute $h_\in$ and we will show that $h_\in$ approaches g uniformly over compact sets in the complex plane when $g(z) = \int_\Omega K(z,t) \, f(t) \, dt$, $z \in C$ and $D\ell = f$ for some $\ell \in L^2(\Omega)$.

<u>Theorem 4</u>

Let K be an integral operator satisfying the properties stated in theorems 2 and 3. Let D, $\widetilde{g}$ and g be as defined above, and let us assume that $f = D\ell$ for a certain $\ell \in L^2(\Omega)$. If we take $\Delta_N = \frac{2a}{2N+1}$, for all $N \geqslant 1$, and if we define

$$h_N(z) = \Delta_N \sum_{j=-N}^{N} \gamma_j \, KD \, \overline{K}(j\Delta_N, \cdot) \, (z) \, , \quad z \in C \tag{36}$$

where γ_j, $j = -N, \ldots, N$ is the optimum of the following problem:

$$\text{minimize} \sum_{j=-N}^{N} \gamma_j \cdot \beta_j \tag{37a}$$

subject to:

$$\Delta_N \sum_{j=-N}^{N} \gamma_j \; KD \; \overline{K}(j\Delta_N, \cdot) \; (i\Delta_N) = \beta_i \;, \quad -N \leq i \leq N \tag{37b}$$

$$\left| \beta_i - \widetilde{g}(i\Delta_N) \right| \leq \in \;, \quad -N \leq i \leq N \tag{37c}$$

Then, there exists a subsequence Δ_{N_k} : $h_{\Delta_{N_k}}$ approaches $h_\in$ uniformly over compact sets in the complex plane, where $h_\in$ is defined by conditions (35a), (35b) and (35c).

Proof

We will follow the same ideas as those of theorem 3. It is clear that h is well-defined because condition (35b) defines a closed convex subset of $L^2(\Omega)$. It is also clear that h_{Δ_N} is well-defined and that condition (37c) can be rewritten as

$$-N \leq i \leq N \;: \; \left| h_{\Delta_N}(i\Delta_N) - g(i\Delta_N) \right| \leq \in \;.$$

It is also possible to rewrite

$$\sum_{j=-N}^{N} \beta_j \gamma_j$$

as follows:

$$\sum_{j=-N}^{N} \beta_j \gamma_j = \sum_{j=-N}^{N} \sum_{k=-N}^{N} \gamma_j \gamma_k \int_\Omega K(j\Delta_N, t) \; D\left(\overline{K}(k\Delta_N, \cdot) \right)(t) \; dt$$

$$= \int_\Omega \sum_{j=-N}^{N} \gamma_j \; K(j\Delta_N, t) \cdot D\left(\sum_{k=-N}^{N} \gamma_k \overline{K}(k\Delta_N, \cdot) \right)(t) \; dt$$

$$= \left\langle \sum_{j=-N}^{N} \gamma_j K(j\Delta_N, \cdot) \;, \; \sum_{j=-N}^{N} \gamma_j K(j\Delta_N, \cdot) \right\rangle_D$$

and therefore, the optimization problem can be restated by using lemma 2 as follows

$$\text{minimize } \langle s, s \rangle_D \tag{38a}$$

subject to

$$s \in L^2 : \; -N \leq i \leq N , \; \left| \langle s, \Delta_N^{\frac{1}{2}} \overline{K}(i\Delta_N, \cdot) \rangle_D - g(i\Delta_N) \right| \leq \epsilon \tag{38b}$$

It is also important to notice that h_{Δ_N} can be wirtten in the following way:

$$h_{\Delta_N}(z) = \Delta_N^{\frac{1}{2}} \int_\Omega S_N^\circ(t) \cdot D\left[K(z, \cdot) \right](t) \, dt \tag{39}$$

where S_N° is the optimum solution to the problem (38a), (38b) which is given by the formula

$$S_N^\circ(t) = \Delta_N^{\frac{1}{2}} \sum_{j=-N}^{N} \gamma_j^\circ \overline{K}(j\Delta_N, t) , \quad t \in \Omega$$

By means of (39), we obtain

$$\left| h_{\Delta_N}(z) \right| \leq \Delta_N^{\frac{1}{2}} \cdot \| S_N^\circ \|_2 \cdot \| D \overline{K}(z, \cdot) \|_2 \tag{40}$$

Since $f = D\ell$, $\ell \in L^2(\Omega)$, then $\Delta^{-\frac{1}{2}}\ell$ also satisfies condition (38b). From (40) and the other properties assumed for D and K we conclude

$$\left| h_{\Delta_N}(z) \right| \leq \left(\int_\Omega \ell(t) \, D\ell(t) \, dt \right)^{\frac{1}{2}} \cdot \| D \|_2 \cdot \frac{1}{m^{\frac{1}{2}}} \cdot C ,$$

where

$$C_\Gamma = \sup_{z \in \Gamma} \left(\int_\Omega |K(z,t)|^2 \, dt \right)^{\frac{1}{2}} , \quad \text{a compact set in C.}$$

We now apply lemma 1 to the sequence h_{Δ_N}, N 0 to get an analytic function

$g_\in$ which is a uniform limit on compact sets of a certain subsequence $h_{\Delta_{N_k}}$, $k \geqslant 0$. It is clear that $g_\in$ satisfies $|g_\in(x) - \tilde{g}(x)| \leqslant \in$ for all $x \in A$ because h_{Δ_N} is an equicontinuous family, $\left\{ i\Delta_N, -N \leqslant i \leqslant N, \Delta_N = \dfrac{2a}{2N+1} \right\}$ is a dense subset of A and η is continuous. Therefore $g_\in$ satisfies (35b).

Since

$$h_{\Delta_{N_k}}(z) = \Delta_{N_k}^{\frac{1}{2}} \int_\Omega S_{N_k}^\circ(t) \cdot D\Big[K(z,\cdot)\Big](t)\, dt,$$

and

$$\| \Delta_N^{\frac{1}{2}} S_N^\circ \|_2 \leqslant c, \quad N \geqslant 1$$

then

$$g_\in(z) = \int_\Omega (DS_\in^\circ)(t)\, K(z,t)\, dt ,$$

where S° is the weak limit in $L^2(\Omega)$ of a certain subsequence of $\Delta_{N_k}^{\frac{1}{2}} \cdot S_{N_k}^\circ$. This means that $g_\in$ satisfies equation (35a).

Since $S_\in^\circ$ is a weak limit of $\Delta_{N_k^m}^{\frac{1}{2}} S_{N_k^m}^\circ$, $k \geqslant 0$, N_k^m a subsequence of N_k, then

$$\int_\Omega S_\in^\circ(t)\, \overline{DS_\in^\circ(t)}\, dt \leqslant \sup_k \int_\Omega \Delta_{N_k^m} S_{N_k^m}^\circ(t)\, \overline{DS_{N_k^m}^\circ(t)}\, dt \tag{41}$$

In addition, by definition of S_N°,

$$\Delta_N \int_\Omega S_N^\circ(t)\, \overline{DS_N^\circ(t)}\, dt \leqslant \int_\Omega S_\in(t)\, \overline{DS_\in(t)}\, dt , \quad \text{for all } N \geqslant 0.$$

and by (41) we obtain

$$\int_\Omega S_\in^\circ(t)\, \overline{DS_\in^\circ(t)}\, dt \leqslant \int_\Omega S_\in(t)\, \overline{DS_\in(t)}\, dt \tag{42}$$

By (42) and (35c) we conclude

$$\int_{\Omega} S^{\circ}_{\in}(t) \overline{DS^{\circ}_{\in}(t)} \, dt \;=\; \int_{\Omega} S_{\in}(t) \overline{DS_{\in}(t)} \, dt$$

Since $S^{\circ}_{\in}$ satisfies condition (35b) then we have

$$S^{\circ}_{\in} \;=\; S_{\in} \quad \text{a.e. in } \Omega.$$

Hence, $h_{\in}(z) = g_{\in}(z)$ for all z and therefore $h_{\in}$ is given as the limit of $h_{\Delta_{N_k}}$, uniformly over compact sets in C.

It can be proved that there exists a sequence of positive numbers $\in_n \to 0$ such that $h_{\in_n} \to g$ uniformly over compact sets in the complex plane. The proof can be done by means of similar arguments to those of theorem 3 and theorem 4 and it will not be included here.

To finish this paper, we would like to point out that efficient algorithms for solving the optimization problem (37a) - (37b) - (37c) are being studied. We hope to present these numerical results in a future paper.

ACKNOWLEDGMENTS

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.455945283174515e+00   .402900397777557e+00
.372156322002411e+00   .369854956865311e+00
.287395805120468e+00   .336150497198105e+00
.202861919999123e+00   .302210360765457e+00
.119716331362724e+00   .268447339534760e+00
.390648767352104e-01   .235255569219589e+00

a   b

Table 1

-.424766153097153e+00
-.444271028041840e+00
-.458988606929779e+00
-.468220114707947e+00
-.471424937248230e+00
-.468059182167053e+00
-.457717299461365e+00
-.440061211585999e+00
-.414908528327942e+00
-.382150769233704e+00
-.341848015785217e+00
-.294149160385132e+00
-.239409029483795e+00
-.178038656711578e+00
-.110701113939285e+00
-.380160398781300e-01

-.381040573120117e-01
-.110723495483398e+00
-.178102493286133e+00
-.239418029785156e+00
-.294158935546875e+00
-.341795921325684e+00
-.382068634033203e+00
-.414759159088135e+00
-.439832687377930e+00
-.457384109497070e+00
-.467596530914307e+00
-.470789909362793e+00
-.467403173446655e+00
-.457928061485291e+00
-.442981123924255e+00
-.423175871372223e+00

-.428087711334229e+00
-.446892082691193e+00
-.461000204086304e+00
-.469755798578262e+00
-.472560673952103e+00
-.468890130519867e+00
-.458308249711990e+00
-.440481245517731e+00
-.415189236402512e+00
-.382336586713791e+00
-.341959029436111e+00
-.294229149818420e+00
-.239458024501801e+00
-.178095579147339e+00
-.110725969076157e+00
-.380620248615742e-01

-.380620248615742e-01
-.110725969076157e+00
-.178095579147339e+00
-.239458024501801e+00
-.294229149818420e+00
-.341959029436111e+00
-.382336586713791e+00
-.415189236402512e+00
-.440481245517731e+00
-.458308249711990e+00
-.468890130519867e+00
-.472560673952103e+00
-.469755798578262e+00
-.461000204086304e+00
-.446892082691193e+00
-.428087711334229e+00

c

d

Table 1

.181424632668495e+00
.297010779380798e+00
.415605872869492e+00
.528099000453949e+00
.625602424144745e+00
.700411260128021e+00
.746863901615143e+00
.761998653411865e+00
.745929777622223e+00
.701887905597687e+00
.635918140411377e+00
.556261599063873e+00
.472481161355972e+00
.394434362649918e+00
.331202089786530e+00
.290093839168549e+00
.275844871997833e+00
.290093839168549e+00
.331202089786530e+00
.394434362649918e+00
.472481161355972e+00
.556261599063873e+00
.635918140411377e+00
.701887905597687e+00
.745929777622223e+00
.761998653411865e+00
.746863901615143e+00
.700411260128021e+00
.625602424144745e+00
.528099000453949e+00
.415605872869492e+00
.297010779380798e+00
.181424632668495e+00

e

Table 1

a	b	c
.196625769138336e+01		
.187897312641144e+01		
.176751244068146e+01	.169285908341408e+00	.169242218136787e+00
.163511228561401e+01	.351469308137894e+00	.351432770490646e+00
.148557937145233e+01	.542826056480408e+00	.542796492576599e+00
.132318949699402e+01	.739235699176788e+00	.739212274551392e+00
.115256929397583e+01	.936341047286987e+00	.936323225498199e+00
.978570222854614e+00	.112966644763947e+01	.112965250015259e+01
.806133329868317e+00	.131474101543427e+01	.131473088264465e+01
.640151381492615e+00	.148722636699677e+01	.148721837997437e+01
.485332340002060e+00	.164303648471832e+01	.164303123950958e+01
.346066176891327e+00	.177846109867096e+01	.177845788002014e+01
.226302295923233e+00	.189027214050293e+01	.189027047157288e+01
.129436030983925e+00	.197582149505615e+01	.197582042217255e+01
.582146234810352e-01	.203312158584595e+01	.203312158584595e+01
.146575104445219e-01	.206091284751892e+01	.206091237068176e+01
-.111310136219345e-06		
.146575104445219e-01	.202678513526917e+01	.202678632736206e+01
.582146234810352e-01	.205869960784912e+01	.205870103836060e+01
.129436030983925e+00	.206091046333313e+01	.206091237068176e+01
.226302295923233e+00	.203311848640442e+01	.203312158584595e+01
.346066176891327e+00	.197581624984741e+01	.197582042217255e+01
.485332340002060e+00	.189026439189911e+01	.189027047157288e+01
.640151381492615e+00	.177844893932343e+01	.177845788002014e+01
.806133329868317e+00	.164301872253418e+01	.164303123950958e+01
.978570222854614e+00	.148720228672028e+01	.148721837997437e+01
.115256929397583e+01	.131470859050751e+01	.131473088264465e+01
.132318949699402e+01	.112962400913239e+01	.112965250015259e+01
.148557937145233e+01	.936287164688110e+00	.936323225498199e+00
.163511228561401e+01	.739167749881744e+00	.739212274551392e+00
.176751244068146e+01	.542742192745209e+00	.542796492576599e+00
.187897312641144e+01	.351367205381393e+00	.351432770490646e+00
.196625769138336e+01	.169163316488266e+00	.169242218136787e+00

Table 2

```
.430034935474396e+00
.480462282896042e+00
.531662702560425e+00
.583049893379211e+00
.634011387825012e+00
.683917999267578e+00
.732135176658630e+00
.778034687042236e+00
.821004986763000e+00
.860463500022888e+00
.895866453647614e+00
.926720023155212e+00
.952588915824890e+00
.973105013370514e+00
.987974286079407e+00
.996982574462891e+00
.100000000000000e+01
.996982574462891e+00
.987974286079407e+00
.973105013370514e+00
.952588915824890e+00
.926720023155212e+00
.895866453647614e+00
.860463500022888e+00
.821004986763000e+00
.778034687042236e+00
.732135176658630e+00
.683917999267578e+00
.634011387825012e+00
.583049893379211e+00
.531662702560425e+00
.480462282896042e+00
.430034935474396e+00
```

a

Table 3

b	c	d
$-.438608229160309e-03$	$-.522017478942871e-03$	$.973609974607825e-03$
$.300382077693939e-02$	$.299024581909180e-02$	$.411226507276297e-02$
$.887742638587952e-02$	$.892090797424316e-02$	$.973103661090136e-02$
$.174739360809326e-01$	$.175526142120361e-01$	$.181229598820210e-01$
$.290687084197998e-01$	$.291705131530762e-01$	$.295510105788708e-01$
$.438877940177917e-01$	$.440077781677246e-01$	$.442402735352516e-01$
$.621192455291748e-01$	$.622414350509644e-01$	$.623708218336105e-01$
$.838932991027832e-01$	$.840104818344116e-01$	$.840708538889885e-01$
$.109291434288025e+00$	$.109402537345886e+00$	$.109411068260670e+00$
$.138320922851563e+00$	$.138423681259155e+00$	$.138399809598923e+00$
$.170929163694382e+00$	$.171010300517082e+00$	$.170979470014572e+00$
$.206990271806717e+00$	$.207060426473618e+00$	$.207024037837982e+00$
$.246320739388466e+00$	$.246372342109680e+00$	$.246338576078415e+00$
$.288645297288895e+00$	$.288688540458679e+00$	$.288658857345581e+00$
$.333646059036255e+00$	$.333667993545532e+00$	$.333654344081879e+00$
$.380921900272369e+00$	$.380940198898315e+00$	$.380930572748184e+00$
$.380931377410889e+00$	$.380945205688477e+00$	$.380930572748184e+00$
$.333647251129150e+00$	$.333673477172852e+00$	$.333654344081879e+00$
$.288647651672363e+00$	$.288692474365234e+00$	$.288658857345581e+00$
$.246314048767090e+00$	$.246390342712402e+00$	$.246338576078415e+00$
$.206980824470520e+00$	$.207093238830566e+00$	$.207024037837982e+00$
$.170902013778687e+00$	$.171067237854004e+00$	$.170979470014572e+00$
$.138277649879456e+00$	$.138515472412109e+00$	$.138399809598923e+00$
$.109225153923035e+00$	$.109549522399902e+00$	$.109411068260670e+00$
$.837982892990112e-01$	$.842289924621582e-01$	$.840708538889885e-01$
$.619850158691406e-01$	$.625529289245605e-01$	$.623708218336105e-01$
$.437070727348328e-01$	$.444283485412598e-01$	$.442402735352516e-01$
$.288296937942505e-01$	$.297331809997559e-01$	$.295510105788708e-01$
$.171696543693542e-01$	$.182796716690063e-01$	$.181229598820210e-01$
$.849115848541260e-02$	$.983729958534241e-02$	$.973103661090136e-02$
$.253264605998993e-02$	$.413399934768677e-02$	$.411226507276297e-02$
$-.101298093795776e-02$	$.874757766723633e-03$	$.973609974607825e-03$

Table 3
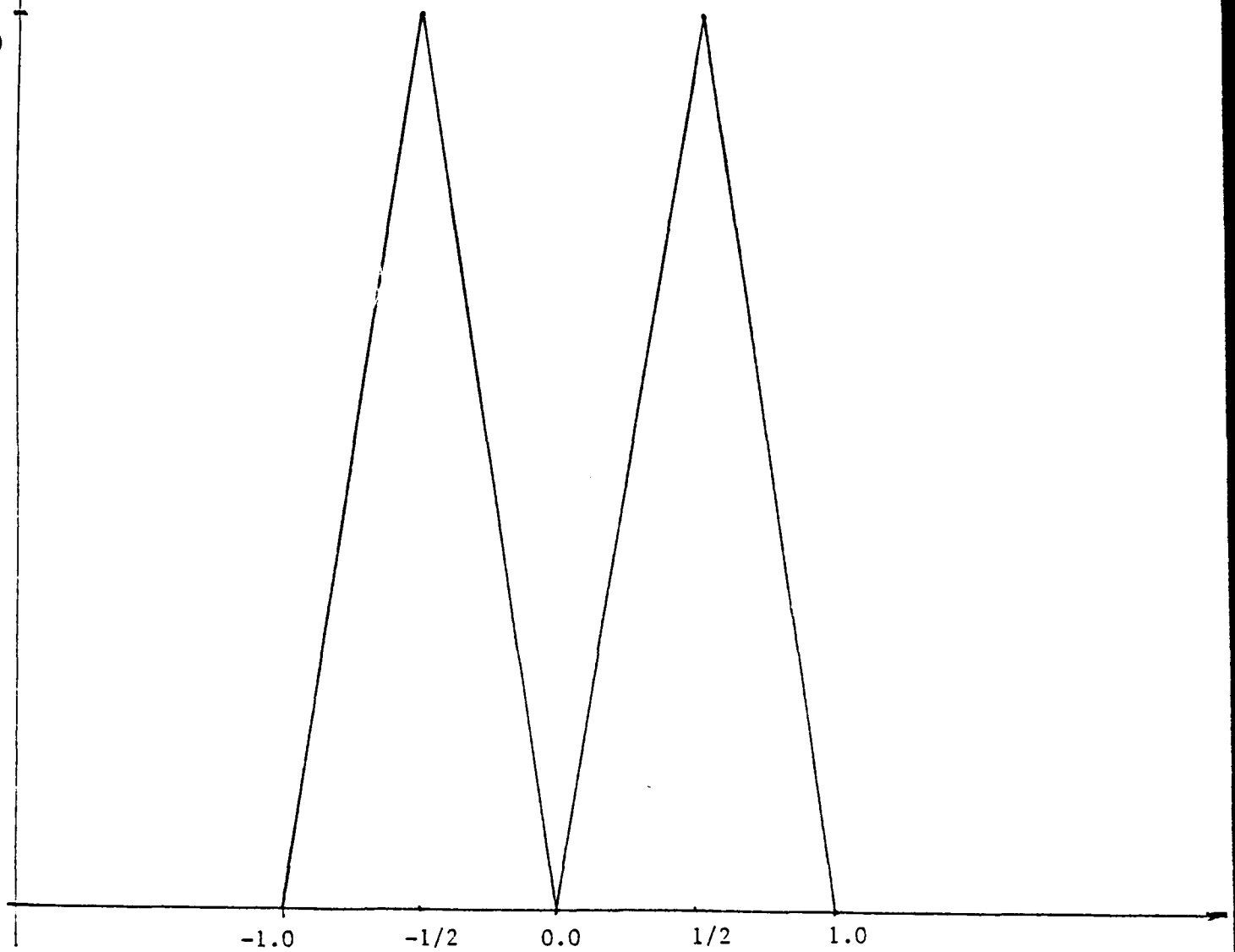
Figure 1.a: Fourier transform of the ideal signal g_1 .
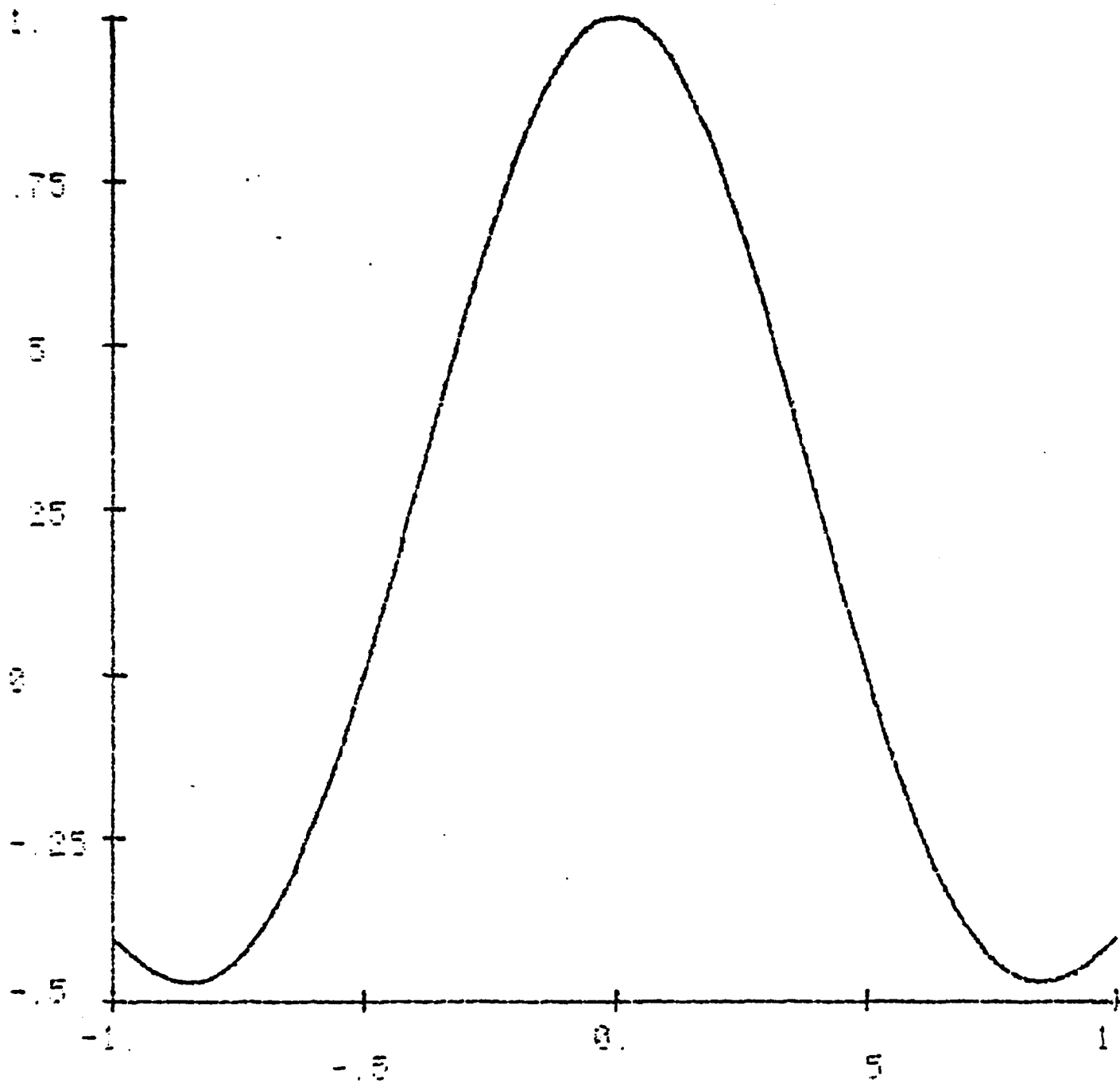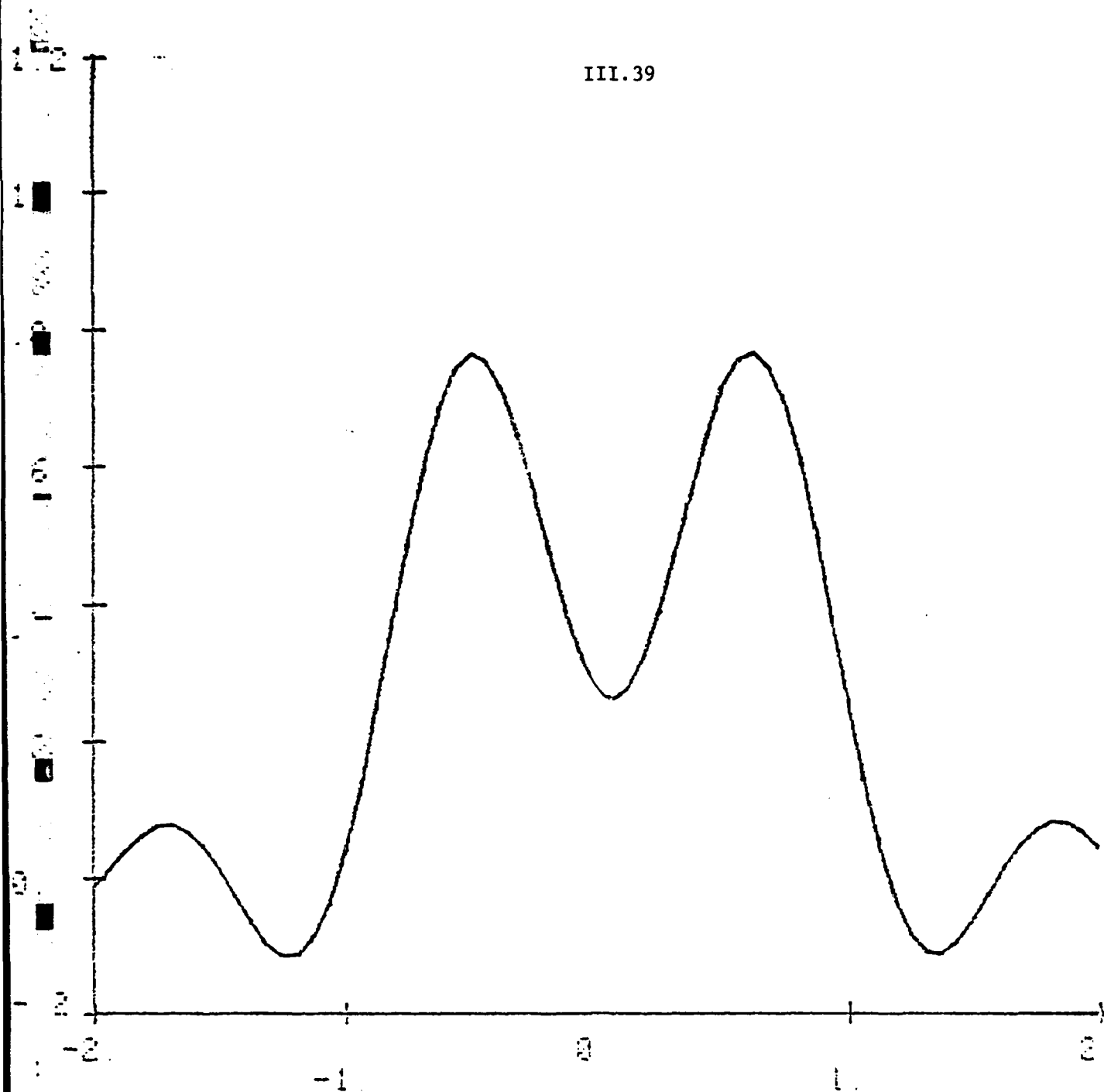
Figure 1.b

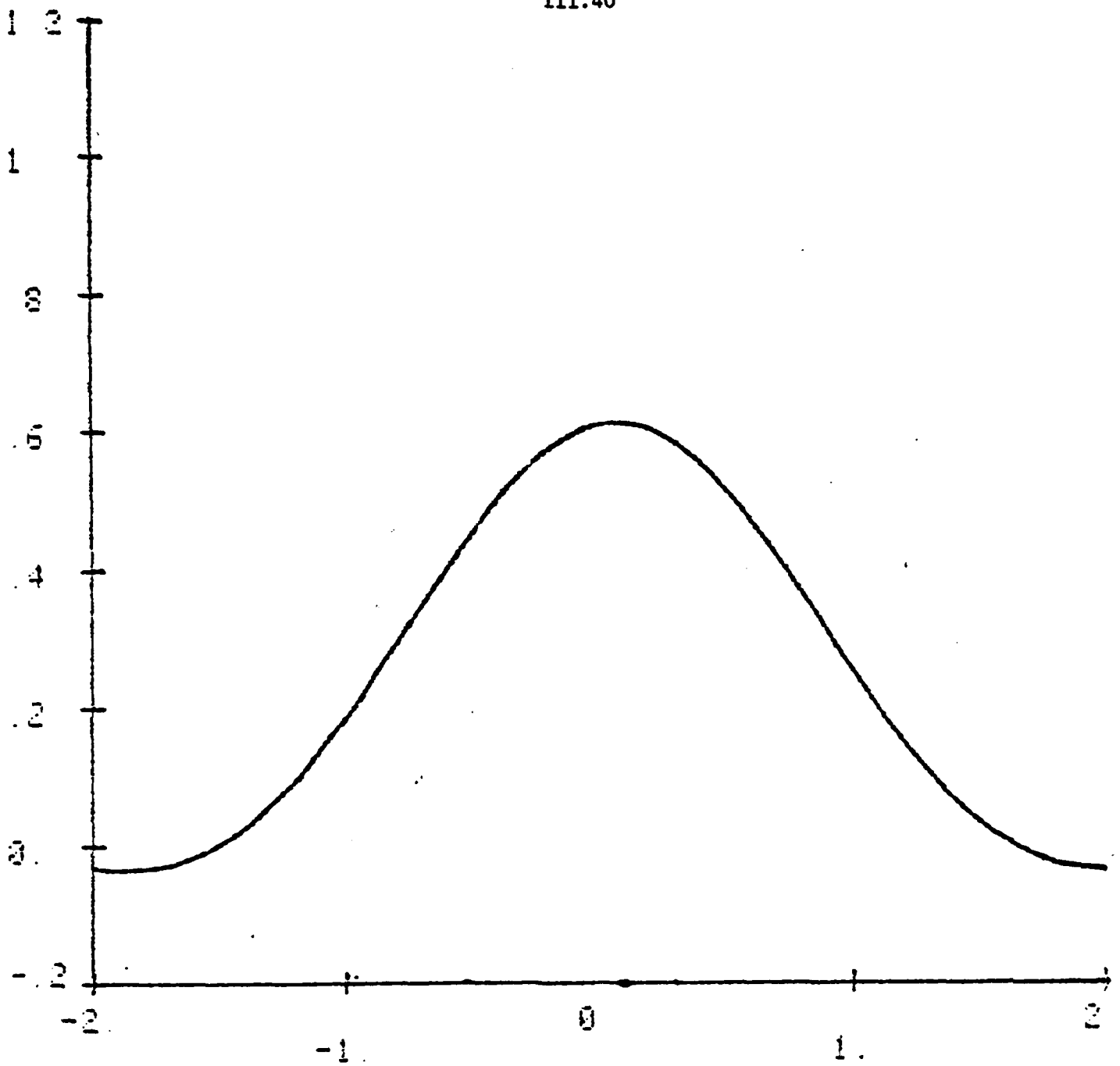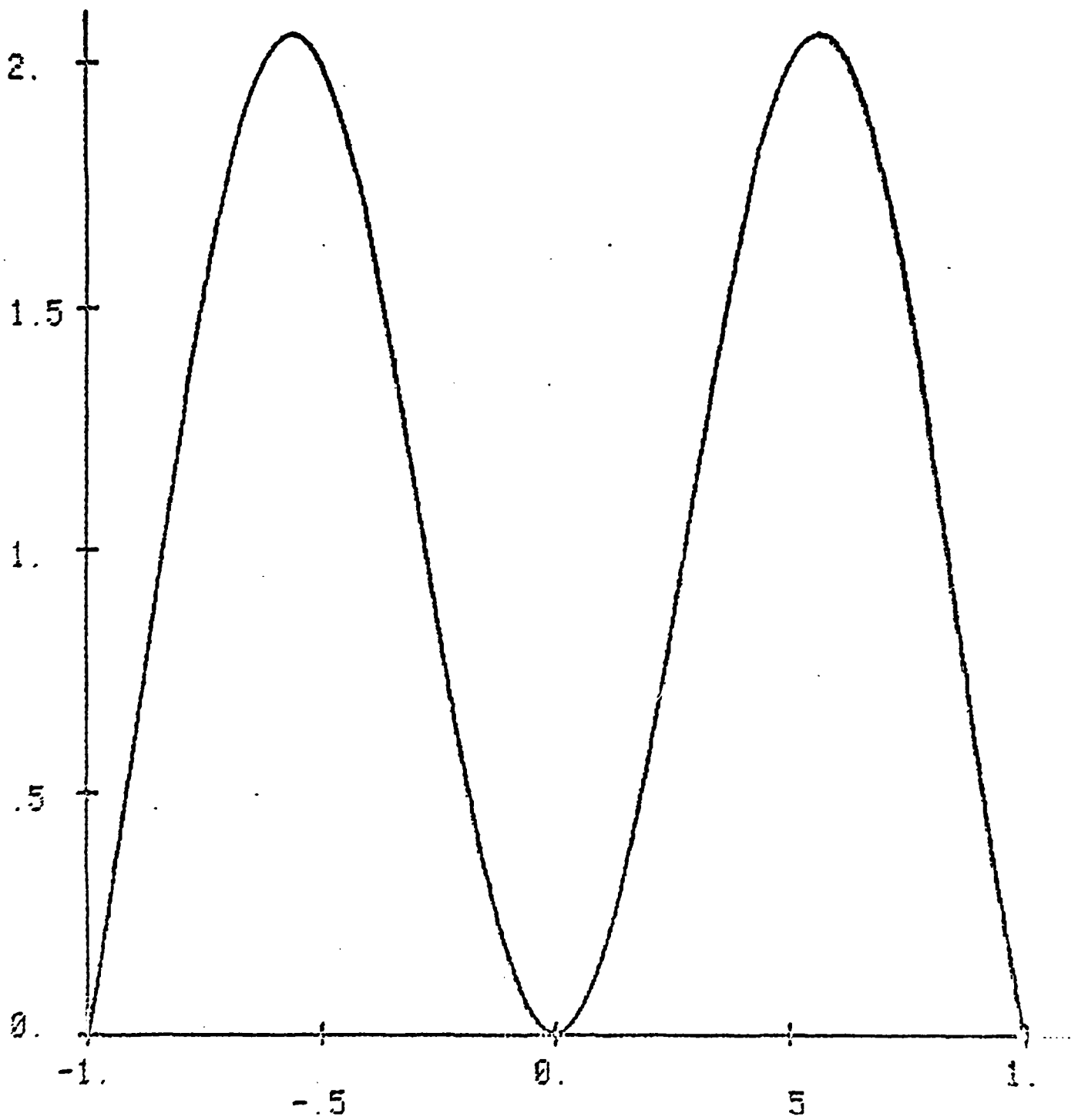Figure 2.a.   DFT of 33 samples in (-1.1)

Figure 2.b.   DFT of 33 samples in (-1/2,1/2)

Figure 3

III.42

Figure 4

4 - 8

DT